DMI College of Engineering -Palanchur
MA3151 –MATRICES AND CALCULUS

Unit-1 - MATRICES
PART- A

Unit-1/ Matrices

A **matrix** is simply a set of numbers arranged in a rectangular pattern.

Types of Matrices:

A matrix having only one row is called a row matrix. Thus A = $[a_{ij}]_{mxn}$ is a row matrix if m = 1.

Ex. A = [1 2 4 5] is row matrix of order 1 x 4.

A matrix having only one column is called a column matrix. Thus,
A = $[a_{ij}]_{mxn}$ is a column matrix if n = 1.

Ex. $A = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ **is column matrix of order 3 x 1.**

If in a matrix all the elements are zero then it is called a zero matrix

Ex $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

If in a matrix there is only element then it is called singleton matrix.

Ex $A = [2], B = [a]$,

If the number of rows and the number of columns in a matrix are equal, then it is called a square matrix.

Eigenvalues and Eigenvectors:

Linear equations Ax= b come from steady state problems. Eigen values have their greatest importance in dynamic problems. This chapter enters a new part of linear algebra, based on Ax= λ x. All matrices in this chapter are square. Where λ is a eigen values and x is a eigen vectors. The Characteristic equation isdet$|A - \lambda I| = 0$ involves only λ not in x. Its roots are called eigenvalues and $(A - \lambda I)X = 0$ corresponding eigenvectors.

Properties:

- Let A be a K x K square matrix. A scalar λ is an eigenvalue of A have the same eigen value of AT.
- The diagonal elements of a triangular matrix are equal to its eigenvalues.
- Sum of the eigen values(Trace of A) = sum of the main diagonal elements
- Product of the eigen values = Determinant of A= |A|
- Eigen values of A = Diagonal elements of triangular matrix(Upper or Lower)

- **If $\lambda_1, \lambda_2 \lambda_3 \dots\dots\dots.\lambda_n$ are eigenvalues of an n×n matrix A, then $\lambda_1^n, \lambda_2^n, \lambda_3^n, \dots\dots\dots..\lambda_n^n$ are the eigenvalues of A^n.**

- **Cayley – Hamilton Theorem:**

  Every squarematrix statistics its own characteristic equation.

1. **Find the sum and product of the eigen values of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.**

Sol:

Sum of the eigen values = sum of the main diagonal elements= 1+2+3 =6

Product of the Eigen valves = |A|

=1(2) – 1 (1)+1(0)

=1

2. **The characteristic equation of a matrix A is $\lambda^2 - 2 = 0$, whit is A^3?**

Sol:

The char. equation of A is

$\lambda^2 - 2 = 0$

$A^2 - 2I = 0$ [By CHT]

Premultiply by A

$A^3 - 2A = 0 \Rightarrow A^3 = 2A$

3. **If 2, 3 are eigen values of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ then find the third eigen value.**

Sol:

Given $\lambda_1 = 2, \lambda_2 = 3$

By Property,

Sum of the eigen values = Sum of diagonal elements

$\lambda_1 + \lambda_2 + \lambda_3 = 3 - 3 + 7$

$2 + 3 + \lambda_3 = 3 - 3 + 7 = 7$

$\boxed{\lambda_3 = 2}$

4. Given $\begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$**, find the eigen values of A^2.**

Sol:

By Property,

Eigen value of A = Diagonal elements of triangular matrix

Hence the eigen values A are -1, -3, 2

By Property,

Eigen value of A^2 are $(-1)^2$, $(-3)^2$, $(2)^2$, 1, 9, 4.

5. Find the sum and product of the eigen values of matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$**.**

Sol:

Sum of the eigen values = 2+3 –6= –1

Product of the eigen values = |A|

=2 (–19) –1(–8)+2(–5)

= – 40

6. Write the matrix of the Quadratic form $4x^2 + 2y^2 - 3z^2 + 2xy + 4xz$

Solution

Matrix of the Quadratic form $\begin{pmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -3 \end{pmatrix}$.

7. If $\lambda_1, \lambda_2 \lambda_3 \ldots \ldots \ldots . \lambda_n$ are eigenvalues of an n×n matrix A, then show that $\lambda_1^3, \lambda_2^3, \lambda_3^3, \ldots \ldots \ldots . . \lambda_n^3$ are the eigenvalues of A^3.

Solution

By defn $Ax_r = \lambda_r \lambda_r$ ....①

Premulting by A in ①

$A(A^2 x_r) = A(\lambda_r x_r)$ $\quad\because \lambda_r$ is a eigin values

$A(Axr) = \lambda_r (Ax_r)$ by①

$$A^2 x_r = \lambda_r (\lambda_r \lambda_r)$$

$$A^2 x_r = \lambda_r^2 x_r$$

$$A^3 x_r = \lambda_r^3 x_r$$

8. **The product of two eigenvalues of the matrix** $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **is 16.**

Find the 3rd eigen value.

Solution

Product of the eigenvalues of $A = |A|$

Given $\lambda_1\lambda_2 = 16$ (ic) $\lambda_1\lambda_2\lambda_3 = |A|16\lambda_3 = 6(9-1)+2(-6+2)+2(2-6)$

$$|6\lambda_3 = 32$$

$$\lambda_3 = 2$$

9. **Verify cayley's – Hamilton theorem for** $A = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$

Solution

$$|A - \lambda I| = 0 \Rightarrow \lambda^2 - 8\lambda + 14 = 0$$

$$A^2 - 8A + 14I = \begin{pmatrix} 10 & -8 \\ -8 & 26 \end{pmatrix} - \begin{pmatrix} 24 & -8 \\ -8 & 40 \end{pmatrix} + \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

10. **Determine the nature of the Quadratic form** $\mathbf{2x^2 + 2y^2 - 3z^2 + 2xy - 4xz + 2yz.}$

Solution

The matrix of the Q.F is $\begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 1 \\ -2 & 1 & -3 \end{pmatrix}$

$$D_1 = |2| > 0$$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 4 - 1 = 3 > 0$$

$$D_3 = \begin{vmatrix} 2 & 1 & -2 \\ 1 & 2 & 1 \\ -2 & 1 & -3 \end{vmatrix} = 2(-6-1) - 1(-3+2) - 2(1+4)$$

$$2(-7) - 1(-1) - 2(5)$$

$$14 + 1 - 10$$

$$-23 < 0$$

$\therefore$ The given Quadratic form is indefinite.

11. If the sum of two eigen values and trace of 3×3 matrix A are equal. Find the value of |A|.

Solution

Let $\lambda_1, \lambda_2, \lambda_3$ he the eigen values of the matrix A

Given:

Sum of two eigen value = Trace of the matrix A

$$\lambda_1+\lambda_2=\lambda_1+\lambda_2+\lambda_3$$

$$\lambda_3 = 0$$

Product of the eigenvalues $|A| = \lambda_1\lambda_2\lambda_3 = 0 \Longrightarrow |A|0$.

12. Find the characteristic ruots of the orthogonal matrix $\begin{pmatrix} cos\theta & -sin\vartheta \\ sin\vartheta & cos\vartheta \end{pmatrix}$ and verify that they are of unit modules.

Solution

$$A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\vartheta \end{pmatrix}$$

The Char. Equation is $|A -\lambda I| = 0$

$$\begin{vmatrix} cos\theta -\lambda & -sin\vartheta \\ sin\vartheta & cos -\lambda \end{vmatrix} = 0 \; \lambda^2 - 2\lambda\, cos\theta + 1 = 0$$

$$\lambda = 2cos\theta \pm \frac{\sqrt{4105^2\theta - 4}}{2} = cos\theta \pm isin\theta$$

∴The Char roots are $cos\theta \pm isin\theta |\lambda| = |cos\theta \pm isin\theta| = \sqrt{cos^2\theta + sin^2\theta} = 1$.

13. Prove that two similar matrices have the same eigen value.

Solution

Let A and B be two similar matrices. Then by definition $B = P^{-1}AP$

$$B -\lambda I = P^{-1}AP -\lambda I$$

$$P^{-1}AP - P^{-1}\lambda IP$$

$$P^{-1}(A -\lambda I)P$$

$$|B -\lambda I| = |P^{-1}||A -\lambda I||P|$$

$$|A -\lambda I||P^{-1}P|$$

$$|A -\lambda I|$$

Thus A and B have the same characteristic equations.

∴ A and B have the same eigen values.

14. If $\begin{pmatrix} \mathbf{4} \\ \mathbf{1} \end{pmatrix}$ is an eigen vector of the matrix $\begin{pmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}$ then find the corresponding eigen value.

Solution

$$X = \begin{pmatrix} 4 \\ 1 \end{pmatrix} A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

Corresponding eigen vector $(A -\lambda I)X = 0$

$$\begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(5-\lambda)4 + 4 = 0 \quad 20 - 4\lambda + 4 = 0 \quad 4\lambda = 24, \lambda = 6$$

$$4 + (2-\lambda)1 = 0 \quad -\lambda = -4 - 2 = -6, \lambda = 6$$

15. . The product of two Eigen values of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third Eigen value.

Solution:

Let the Eigen value of the matrix A be $\lambda_1, \lambda_2, \lambda_3$

Given $\lambda_1 \lambda_2 = 16$

We know that $\lambda_1 \lambda_2 \lambda_3 = |A|$

$$\therefore \lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1) + 2(-6+2) + 2(2-6) = 6(8) + 2(-4) + 2(-4)$$

$= 48 - 8 - 8 = 32$

$\lambda_1 \lambda_2 \lambda_3 = 32 \Rightarrow 16\lambda_3 = 32 \Rightarrow \lambda_3 = \frac{32}{16} = 2$

16. Two of the Eigen values of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the Eigen values of A^{-1}.

Solution:

Sum of the Eigen value $= 3 + 5 + 3 = 11$

Let k be the third Eigen value

$\therefore 3 + 6 + k = 11$

$9 + k = 11$

$k = 2$

$\therefore$ The Eigen value of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

17. State Cayley – Hamilton theorem.

Statement:

Every square matrix satisfies its own characteristic equation.

18. Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$

Solution:

$$Q=\begin{bmatrix} coeff\ x^2 & \frac{1}{2}coeff\ xy & \frac{1}{2}coeff\ xz \\ \frac{1}{2}coeff\ yx & coeff\ y^2 & \frac{1}{2}coeff\ yz \\ \frac{1}{2}coeff\ zx & \frac{1}{2}coeff\ zy & coeff\ z^2 \end{bmatrix}$$

$$Q=\begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

19. Determine the nature of the following quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$

Solution:

The matrix of the Q. F is

$$Q=\begin{bmatrix} coeff\ x_1^2 & \frac{1}{2}coeff\ x_1x_2 & \frac{1}{2}coeff\ x_1x_3 \\ \frac{1}{2}coeff\ x_2x_1 & coeff\ x_2^2 & \frac{1}{2}coeff\ x_2x_3 \\ \frac{1}{2}coeff\ x_3x_1 & \frac{1}{2}coeff\ x_3x_2 & coeff\ x_3^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$D_1 = |1| = 1 \quad (+\text{ve})$, $D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \quad (+\text{ve})$, $|D_3| = 0$

$\therefore$ The Q. F is said to be positive semi definite. Or one eigen value is zero others positive valus then its positive semi definite

20. . Find the sum of the squares of the Eigen values of $\begin{pmatrix} 1 & 7 & 5 \\ 0 & 2 & 9 \\ 0 & 0 & 5 \end{pmatrix}$

Solution:

Eigen values are 1, 2, and 5

Sum of the squares $1^2 + 2^2 + 5^2 = 30$

21. Write the matrix of the quadratic form $Q(x, y) = 3x^2 + 2y^2 - 4xy$

Solution:

$$A=\begin{pmatrix} 3 & -2 \\ -0 & 2 \end{pmatrix}$$

22. Find the Eigen values of A^2 **, given** $A=\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

Solution:

Eigen values of A are 1 and 3

Eigen values of $A^2 = 1, 9$

23. Find the sum and product of Eigen value of $A=\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 6 & 7 \end{bmatrix}$

Solution:

Sum of Eigen values = 1 + 3 + 7 = 11

Product of Eigen values = $|A| = 8$

24. . Write down the matrix corresponding to the quadratic form

$2x^2 + 2y^2 + 3z^2 + 2xy - 4zx - 4yz$

Solution:

$$Q = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

25. Discuss the nature of quadratic form for $2xy + 2yz + 2zx$

Solution:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } D_1 = 0,\ D_2 = -1 < 0,\ D_3 = 2 > 0$$

It is indefinite in nature.

26. For what of k, the vectors $X = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$**,** $Y = \begin{pmatrix} 2 & -1 & k \end{pmatrix}$ **and** $Z = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$ **are linearly dependent.**

Solution:

$$\text{If } \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & k \\ 1 & 0 & -1 \end{vmatrix} = 0 \Rightarrow 1 - 2(-2 - \text{k}) + 3(1) = 0 \Rightarrow 2\,\text{k} + 8 = 0 \Rightarrow \text{k} = -4$$

If $\text{k} = -4$ the vectors are dependent

27. . If $X = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **is the Eigen vector of a matrix A, find the Eigen vector of** A^{-1}

Solution:

A^{-1} is also having the same Eigen vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

28. Using Cayley – Hamilton theorem, is it possible to find the inverse of all square matrices? Explain.

Solution:

Not possible. [Since inverse is possible only for a non – singular matrix]

29. If 3 and 5 are the two Eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$**, then find** $|A|$

Solution:

$3 + 5 + \lambda = 8 + 7 + 3\lambda = 10$

$\therefore |A| = 3 \times 5 \times 10 = 150$

30. . If one of the Eigen vectors of $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ **is** $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ **then find the corresponding Eigen value.**

Solution:

$A\lambda = \lambda X$ Gives $\lambda = 2$

31. . If the Eigen value of A are 2, 3, and 4. Find the Eigen values of $Adj\ A$.

Solution:

Eigen values of $Adj\ A$ = Eigen values $|A|\,A^{-1}$

$\therefore$ Eigen values of $Adj\ A$ are 12, 8, 6

32. . Find the index, signature and nature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$

Solution:

Index = Number of positive square terms in the canonical (given) form = 2.

Signature = Difference between the number of positive square terms and the negative Square terms in the canonical form = 2 – 1 = 1

Since the Eigen values are 1, 2, -3 (mixed up with positive and negative) the quadratic form is indefinite.

II Differential Calculus

Part-A

1. Find the domain of $f(x) = \sqrt{2 - \sqrt{x}}$.
   Solution:
   $$f(x) \text{ is real if } 2 - \sqrt{x} \geq 0$$
   $$\Rightarrow 2 \geq \sqrt{x}$$
   $$\Rightarrow x \leq 4.$$
   Therefore the domain of $f(x) = [0, 4]$.

2. Find the domain of $f(x) = \dfrac{x^2 - 1}{x + 1}$.
   Solution:
   $$f(x) = \frac{(x+1)(x-1)}{x+1}$$
   $$\Rightarrow f(x) = x - 1$$
   Therefore the domain of $f(x)$ is $(-\infty, \infty)$.

3. Find the domain and range of $f(x) = 2 + \sqrt{x - 1}$.
   Solution:
   $$f(x) \text{ is real if } x - 1 \geq 0$$
   $$\Rightarrow x \geq 1.$$
   Therefore the domain of $f(x)$ is $[1, \infty)$.
   To find the range, let
   $$y = 2 + \sqrt{x - 1}$$
   $$\Rightarrow y - 2 = \sqrt{x - 1}$$
   $$\Rightarrow (y - 2)^2 = x - 1,$$
   a parabola with vertex $(1, 2)$ and symmetric about x-axis.
   Since $y = 2 + \sqrt{x - 1}$, this corresponds to the upper half of the parabola.
   Therefore the range of $f(x)$ is $[2, \infty)$.

4. State vertical line test.
   Solution: A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

5. Determine the nature of the following functions.
   (i) $f(x) = 1 - x^4$ (ii) $f(x) = x^5 - x$
   (iii) $f(x) = 2x - x^2$.
   Solution:
   (i). $f(x) = 1 - x^4$
   $$f(-x) = 1 - (-x)^4 = 1 - x^4 = f(x).$$
   $$\therefore f(x) = 1 - x^4 \text{ is an even function.}$$
   (ii). $f(x) = x^5 - x$
   $$f(-x) = (-x)^5 - (-x) = -x^5 + x$$
   $$= -(x^5 - x) = -f(x).$$
   $$\therefore f(x) = x^5 - x \text{ is an odd function.}$$
   (iii). $f(x) = 2x - x^2$.
   $$f(-x) = 2(-x) - (-x)^2 = -2x - x^2$$
   $$= -(2x + x^2).$$
   $$\therefore f(x) = 2x - x^2 \text{ is neither even nor odd.}$$

6. Evaluate $\lim_{x \to 0} \dfrac{(3 + x)^2 - 9}{x}$.
   Solution:
   $$\lim_{x\to 0} \frac{(3+x)^2 - 9}{x} = \lim_{x\to 0} \frac{(3+x)^2 - 3^2}{x}$$
   $$= \lim_{x\to 0} \frac{(3 + x + 3)(3 + x - 3)}{x}$$
   $$= \lim_{x\to 0} \frac{x(x+6)}{x}$$
   $$= \lim_{x\to 0}(x + 6) = 6.$$

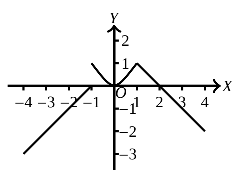
7. Find $\lim_{x \to 1} g(x)$ where
   $$g(x) = \begin{cases} x + 1, & \text{if } x \neq 1 \\ \pi, & \text{if } x = 1 \end{cases}.$$

Solution: At $x = 1, g(x) = \pi$.
But $\lim_{x\to 1} g(x)$ does not depend on the value of $g(x)$ at $x = 1$.

$$\therefore \lim_{x\to 1} g(x) = \lim_{x\to 1}(x+1) = 2.$$

8. Sketch the graph of the function $f(x) = \begin{cases} x+1, & \text{if } x < -1 \\ x^2, & \text{if } -1 \le x \le 1 \\ 2-x, & \text{if } x \ge 1 \end{cases}$ and use it to determine the value of a for which $\lim_{x\to a} f(x)$ exists.
Solution:

From the graph, $\lim_{x\to a} f(x)$ exists for all a except $a = -1$, since the left and right limits at $a = -1$ are different.



9. If $f(x) = \frac{|x|}{x}$, $x \ne 0$, find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.
Solution: Given $f(x) = \frac{|x|}{x}$

$$= \begin{cases} \frac{x}{x}, & \text{if } x \ge 0 \\ \frac{-x}{x}, & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 1, & \text{if } x \ge 0 \\ -1, & \text{if } x < 0 \end{cases}$$

$f(x)$ is not defined at $x = 0$ but defined in the neighborhood of 0.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1.$$

10. State squeeze theorem.(Sandwich theorem/Pinching theorem).
Solution:
If $f(x) \le g(x) \le h(x)$, where x is near a(except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

11. Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.
Proof:

$$\lim_{x\to 0} x^2 sin\left(\frac{1}{x}\right) = \lim_{x\to 0} \frac{x\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to 0} x = 0.$$

$$(\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1)$$

12. If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$ find $\lim_{x\to 4} f(x)$ using squeeze theorem.
Solution: Let $g(x) = 4x - 9$ and

$$h(x) = x^2 - 4x + 7.$$

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} 4x - 9 = 7 \ \&$$

$$\lim_{x\to 4} g(x) = \lim_{x\to 4} x^2 - 4x + 7 = 7$$

We see that $f(x)$ is squeezed between $g(x)$ & $h(x)$ near $x = 4$.
Since $g(x)$ & $h(x)$ have the same limit 7 at $x = 4$, by squeeze theorem $f(x)$ also has the limit 7.

13. Evaluate $\lim_{x\to\infty} \frac{4x^2+5x+6}{3x^2+4x+5}$
Solution:

$$\lim_{x\to\infty} \frac{4x^2+5x+6}{3x^2+4x+5} = \lim_{x\to\infty} \frac{x^2\left(4+\frac{5}{x}+\frac{6}{x^2}\right)}{x^2\left(3+\frac{4}{x}+\frac{5}{x^2}\right)} = \frac{4}{3}$$

14. State intermediate value theorem.
Solution: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \ne f(b)$. Then there exists a number C in (a, b) such that $f(C) = N$.

15. Test the continuity of $f(x) = \frac{x^2+5x}{2x+1}$, at

$x = 2$.

Solution: $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2+5x}{2x+1}$
$= \frac{4+10}{4+1} = \frac{14}{5}$

Also $f(2) = \frac{x^2+5x}{2x+1} = \frac{4+10}{4+1} = \frac{14}{5}$.

Since $\lim_{x\to 2} f(x) = f(2) = \frac{14}{5}$, we see that $f(x)$ is continuous at $x = 2$.

16. Test the continuity at $x = 3$ for

$$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3}, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases}$$

Solution: $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2x^2-5x-3}{x-3}$
$= \lim_{x\to 3} \frac{(2x+1)(x-3)}{x-3}$
$= 7$

and $f(3) = 6$

$\because \lim_{x\to 3} f(x) \neq f(3)$,

$\therefore f(x)$ is discontinuous at $x = 3$.

17. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents ? If so, where ?

Solution: The horizontal tangents occur when $\frac{dy}{dx} = 0$.

Now, $\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2-1) = 0$

$\Rightarrow x = 0$ or $x = \pm 1$.

Therefore the curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1, -1$. The corresponding points on the curve are $(0,2), (1,1)$ and $(-1,1)$.

18. For what value of a is

$f(x) = \begin{cases} x^2-1, & \text{if } x < 3 \\ 2ax, & \text{if } x \geq 3 \end{cases}$ continuous on $(-\infty, \infty)$.

solution:

Let $f(x)$ be continuous everywhere.

$\Rightarrow f$ is continuous at $x = 3$,

$\therefore \lim_{x\to 3} f(x) = f(3)$

$\Rightarrow \lim_{x\to 3+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 - 1 = 8$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2ax = 6a$

$\Rightarrow f(3) = 6a$

$\therefore 6a = 8$

$\Rightarrow a = \frac{4}{3}$.

19. Find the equation of the tangent line to the parabola $y = x^2$ at $(1,1)$.

solution:

Given $a = 1$ and $f(x) = x^2$

$\therefore f(1) = 1$

Slope $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$
$= \lim_{h\to 0} \frac{(1+h)^2-1}{h}$
$= \lim_{h\to 0} \frac{1+h^2+2h-1}{h}$
$= \lim_{h\to 0}(h+2)$
$= 2$.

The equation of the tangent line through $(1,1)$ is

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x-1)$

$\Rightarrow y = 2x - 1$.

20. If $f'(x)$ exists, then $\lim_{x\to a} f(x) = f(a)$. True or false?

Solution: True, since differentiability implies continuity.

21. Give an example of a function which is continuous at 'a' but not differentiable at 'a'.

Solution:

Consider $f(x) = |x|$.

$f(x) = |x|$ is continuous at $x = 0$,

since $\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0$ but is not differentiable at $x = 0$.

22. Evaluate $\lim_{x\to \frac{\pi}{2}} \frac{1+cos2x}{(\pi-2x)^2}$.

Solution:

$$\lim_{x\to\frac{\pi}{2}} \frac{1+cos2x}{(\pi-2x)^2} = \lim_{x\to\frac{\pi}{2}} \frac{2cos^2 x}{(\pi-2x)^2}$$

Let $x = \frac{\pi}{2} + y$ as $x \to \frac{\pi}{2}, y \to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x\to\frac{\pi}{2}} \frac{2\cos^2 x}{(\pi-2x)^2}$$

$$= \lim_{y\to 0} \frac{2\cos^2(\frac{\pi}{2})}{(\pi - 2(\frac{\pi}{2}+y))^2}$$

$$= \lim_{y\to 0} \frac{2\sin^2 y}{4y^2}$$

$$= \frac{1}{2}\lim_{y\to 0}(\frac{\sin y}{y})^2$$

$$= \frac{1}{2}(\lim_{y\to 0}\frac{\sin y}{y})^2$$

$$= \frac{1}{2}.1^2 = \frac{1}{2}.$$

23. If c is a constant, prove that $\frac{d}{dx}(c) = 0$.

Proof: Let $f(x) = c$

$$f'(x) = \lim_{x\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{x\to 0}\frac{c-c}{h} = \lim_{x\to 0}\frac{0}{h} = 0.$$

24. Prove that $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$

Proof:

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} = \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}.\frac{\cos\theta+1}{\cos\theta+1}$$

$$= \lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}$$

$$= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$

$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim_{\theta\to 0}\frac{\sin\theta}{(\cos\theta+1)}$$

$$= -1\times\frac{0}{1+1} = 0$$

$$\therefore \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} = 0.$$

25. At what point on the curve $y = e^x$, is the tangent line parallel to the line $y = 2x$.

Solution: Given $y = e^x \implies y' = e^x$.

Slope of the tangent at $x = e^x$.

Since the tangent line is parallel to the line $y = 2x$, slope of the tangent = 2.

That is $e^x = 2 \implies x = \log 2$.

When $x = \log 2$, $y = e^{\log 2} = 2$.

$\therefore$ The required point is $(\log 2, 2)$.

26. If $\sqrt{x+\sqrt{x+\sqrt{x+\cdots\infty}}}$, find $\frac{dy}{dx}$

Solution:

Given $\sqrt{x+\sqrt{x+\sqrt{x+\cdots\infty}}}$.

Let $y = \sqrt{x+y} \Rightarrow y^2 = x + y$.

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{2y-1}.$$

27. Find $\frac{d}{dx}(\sin x)^{\cos x}$

Solution: Let $y = (\sin x)^{\cos x}$

Take logarithms on both sides, we get

$$\log y = \log(\sin x)^{\cos x}$$

$$= \cos x.\log(\sin x)$$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{1}{\sin x}.\cos x + \log(\sin x).(-\sin x)$$

$$= \frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)$$

$$\therefore \frac{dy}{dx} = y\left[\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right]$$

$$= (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right]$$

28. Find the derivative of

(i). $x\tan^{-1}(\sqrt{x})$ or $x\tan^{-1}\sqrt{x}$

(ii). $(1+x^2)\tan^{-1}(x)$

Solution:

(i). Let $y = x\tan^{-1}(\sqrt{x})$

$$\frac{dy}{dx} = x.\frac{1}{1+(\sqrt{x})^2}.\frac{1}{2\sqrt{x}} + \tan^{-1}(\sqrt{x}).1$$

$$= \frac{\sqrt{x}}{2(1+x)} + \tan^{-1}(\sqrt{x}).$$

(ii). Let $y = (1 + x^2) \tan^{-1} x$

$$\frac{dy}{dx} = (1 + x^2).\frac{1}{1 + x^2} + \tan^{-1} x.2x$$
$$= 1 + 2x \tan^{-1} x.$$

29. If $f(x) = x^{25}(1 - x)^{75}$, $x \in [0, 1]$, find the point at which $f(x)$ is maximum.

Solution: Given $f(x) = x^{25}(1 - x)^{75}$

$$f'(x) = x^{25}.75(1 - x)^{74}(-1) + (1 - x)^{75}.25x^{24}$$
$$= x^{24}(1 - x)^{74}[-75x + 25(1 - x)]$$
$$= x^{24}(1 - x)^{74}[25 - 100x]$$
$$= 25x^{24}(1 - x)^{74}(1 - 4x)$$
$$= -25x^{24}(x - 1)^{74}(4x - 1)$$
$$= -100x^{24}(x - 1)^{74}\left(x - \frac{1}{4}\right).$$

If $x > \frac{1}{4}$, $f'(x) < 0$ and

if $x < \frac{1}{4}$, $f'(x) > 0$

$\therefore$ f has maximum at $x = \frac{1}{4}$.

30. Find the absolute maximum of $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$.

Solution: Given $f(x) = x^{\frac{2}{3}}$

$$\Rightarrow f'(x) = \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3x^{\frac{1}{3}}}.$$

$f'(0)$ does not exist.

$\therefore x = 0$ is a critical point. $\Rightarrow f(0) = 0$.

$$f(-2) = (-2)^{\frac{2}{3}} = \left[(-2)^2\right]^{\frac{1}{3}} = \sqrt[3]{4} \text{ or } 4^{\frac{1}{3}}$$
$$f(3) = (3)^{\frac{2}{3}} = \left(3^2\right)^{\frac{1}{3}} = \sqrt[3]{9} \text{ or } 9^{\frac{1}{3}},$$

which is the absolute maximum.

31. Find the maximum value of the product of the numbers whose sum is 12.

Solution: Let x, y be the numbers such that $x + y = 12 \Rightarrow y = 12 - x$

Let $p = xy$

$$= x(12 - x) = 12x - x^2$$
$$\frac{dp}{dx} = 12 - 2x.$$
$$\frac{dp}{dx} = 0$$
$$\Rightarrow 12 - 2x = 0$$
$$\Rightarrow x = 6$$
$$\frac{d^2p}{dx^2} = -2 < 0$$

$\therefore$ p is maximum at $y = 6$

$\therefore$ The maximum value $= xy = 6(6) = 36$.

32. Determine the critical points of the function $f(x) = x^4 - 2x^2 + 3$

Solution: Given $f(x) = x^4 - 2x^2 + 3$

$$\Rightarrow f'(x) = 4x^3 - 4x$$
$$f'(x) = 0$$
$$\Rightarrow 4x^3 - 4x = 0$$
$$\Rightarrow 4x\left(x^2 - 1\right) = 0$$
$$\Rightarrow 4x(x - 1)(x + 1) = 0$$
$$\Rightarrow x = 0, 1, -1$$

$\therefore$ The critical points are $x = -1$, $x = 0$, $x = 1$.

33. Using first derivative test, examine the maximum and minimum of $f(x) = \sin^2 x$, $0 < x < \pi$.

Solution:

Given $f(x) = \sin^2 x$

$$\Rightarrow f'(x) = 2\sin x \cos x = \sin 2x$$

Critical points are occurs at

$$f'(x) = 0$$
$$\Rightarrow \sin 2x = 0$$
$$\Rightarrow 2x = 0 \text{ or } 2x = \pi$$
$$\therefore x = 0 \text{ or } x = \frac{\pi}{2}$$

$\therefore$ The critical points are $x = 0$ or $x = \frac{\pi}{2}$.

Interval	$\sin 2x$	$f'(x)$	f
$0 < x < \frac{\pi}{2}$	+	+	increases in $\left(0, \frac{\pi}{2}\right)$
$x > \frac{\pi}{2}$	−	−	decreases in $\left(\frac{\pi}{2}, \pi\right)$

Since f' changes from positive to negative at $x = \frac{\pi}{2}$, f has a maximum at $x = \frac{\pi}{2}$.

$\therefore$ Maximum value $f = f\left(\frac{\pi}{2}\right)$

$$= \sin^2\left(\frac{\pi}{2}\right) = 1.$$

UNIT – III FUNCTIONS OF SEVERAL VARIABLES

PART – A

1. **If $u = \frac{y}{z} + \frac{z}{x}$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$**

Solution: Given $u = \frac{y}{z} + \frac{z}{x}$,

$$\frac{\partial u}{\partial x} = -\frac{z}{x^2};\ \frac{\partial u}{\partial y} = \frac{1}{z};\ \frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(-\frac{z}{x^2}\right) + y\left(\frac{1}{z}\right) + z\left(-\frac{y}{z^2} + \frac{1}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x} = 0$$

2. **Find $\frac{du}{dt}$ when $u = x^2 + y^2,\ x = t^3,\ y = t^2$**

Solution:

$$u = x^2 + y^2,\ x = t^3,\ y = t^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{dx}{dt} = 3t^2$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{dy}{dt} = 2t$$

$$\frac{du}{dt} = 2x\left(3t^2\right) + 2y\left(2t\right) \Rightarrow 6xt^2 + 4yt$$

$$\frac{du}{dt} = 6t^3t^2 + 4t^2t = 2t^3(3t^2 + 2) \qquad \left(\because x = t^3 \quad y = t^2\right)$$

3. **If $u = x^3 + y^3$ and $x = at^2,\ y = 2at$, find $\frac{du}{dt}$** (M

Solution:

Given $u = x^3 + y^3$ and $x = at^2,\ y = 2at$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = 3x^2 \qquad \frac{dx}{dt} = 2at$$

$\frac{dx}{dt} = 3y^2$ $\qquad\qquad$ $\frac{dy}{dt} = 2a$

$$\frac{du}{dt} = (3x^2)(2at) + (3y^2)(2a) = (3(\text{at}^2)^2)(2at) + (3)(2\,\text{at})^2(2a) \qquad \left(\because x = at^2,\ y = 2at\right)$$

$$= 6a^3t^5 + 24a^3t^2 = 6a^3t^2(\text{t}^3 + 4)$$

4. If $x^y + y^x = c$, find $\frac{dy}{dx}$

Solution: Let $f\left(x, y\right) = x^y + y^x - c$

$$\frac{\partial f}{\partial x} = = yx^{y-1} + y^x \log y \qquad \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1} \qquad \left(\because \frac{d}{dx}\left(a^x\right) = a^x \log \text{a}\right)$$

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\left(\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$$

5. Find $\frac{dy}{dx}$ if $3x^2 + xy - y^2 + 4x - 2y + 1 = 0$

Solution: Let $f\left(x, y\right) = 3x^2 + xy - y^2 + 4x - 2y + 1$

$$\frac{dy}{dx} = \frac{-\partial f / \partial x}{\partial f / \partial y} = -\left(\frac{6x + y + 4}{x - 2y - 2}\right)$$

6. Find $\frac{du}{dt}$, if $u = xy + yz + zx$ where $x = t$, $y = e^t$, $z = t^2$

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt} = \left(y + z\right)\left(1\right) + \left(x + z\right)\left(e^t\right) + \left(y + x\right)\left(2t\right)$$

$$= 3t^2 + e^t\left(1 + 3t + t^2\right)$$

7. Find $\frac{du}{dt}$, if $u = \log(x^2 + y^2 + z^2)$ where $x = e^t$, $y = \sin t$, $z = \cos t$

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$= \left(\frac{1}{x^2 + y^2 + z^2}\right)(2x)(-e^{-t}) + \left(\frac{1}{x^2 + y^2 + z^2}\right)(2y)(\cos t) + \left(\frac{1}{x^2 + y^2 + z^2}\right)(2z)(-\sin t)$$

$$= \left(\frac{1}{x^2 + y^2 + z^2}\right)\left[(x(-x) + yz - zy\right]$$

$$= \left(\frac{-2e^{-2t}}{e^{-2t} + \sin^2 t + \cos^2 t}\right) = \left(\frac{-2e^{-2t}}{e^{-2t} + 1}\right)$$

8 State the properties of Jacobian.

Solution:

(i) If u and v are functions of r and s , r and s are functions of x and y then,

$$\frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

(ii) If u and v are functions of x and y then, $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$ (i.e) $JJ' = 1$

(iii) If u,v,w are functionly dependent functions of three independent variable x,y,z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

9. **If $x = u^2 - v^2$ and $y = 2uv$, find the jacobian of x and y with respect to u and v.**

Solution: $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4(u^2 + v^2)$$

10 **If $x = uv$ and $y = \frac{u}{v}$ find $\frac{\partial(x,y)}{\partial(u,v)}$**

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$= v\left(\frac{-u}{v^2}\right) - u\left(\frac{1}{v}\right) = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v}$$

11. **If $x = r\cos\theta,\ y = r\sin\theta$ *then find* $\frac{\partial(r,\theta)}{\partial(x,y)}$**

Solution:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}} = \frac{1}{r}$$

12. **If $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$, find $\frac{\partial(u,v)}{\partial(x,y)}$**

Solution: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

13. **If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\cot u$**

Solution: $\mathbf{u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}}$

Cos u is aa homogeneous function of degree $\mathbf{\frac{1}{2}}$ in x and y.

Therefore, by Euler's theorem we get,

$$x\frac{\partial(\cos u)}{\partial x} + y\frac{\partial(\cos u)}{\partial y} = \frac{1}{2}\cos u \rightarrow (1)$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$-\sin u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u$$

14. **Find Taylor's series expansion of x^y near the point (1,1) up to first degree terms.**

Solution:

$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \ldots$

$f(x,y) = x^y$, $\quad f(1,1) = 1$

$f_x(x,y) = yx^{y-1}$, $\quad f_x(1,1) = 1$

$f_y(x,y) = x^y \log x$, $\quad f_y(1,1) = 0$

$f(x,y) = 1 + (x-1)(1) + (y-1)(0) + \ldots$

$x^y = x + \ldots$

15. **Obtain Taylor's series expansion of e^{x+y} in powers of x and y upto first degree terms.**

Solution:

$f(x,y) = e^{x+y}$, $\quad f_x(x,y) = e^{x+y}$, $\quad f_y(x,y) = e^{x+y}$

$f(0,0) = 1$, $\quad f_x(0,0) = 1$, $\quad f_y(0,0) = 1$.

$\therefore f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \ldots$

At (0,0), $\quad f(x,y) = f(0,0) + (x)f_x(0,0) + (y)f_y(0,0) + \ldots$

$e^{x+y} = 1 + x + y + \ldots$

16. **Expand $e^x \cos y$ in Taylor's series in powers of x and y up to terms of first degree.**

Solution:

$f(x,y) = e^x \cos y$, $\quad f(0,0) = 1$

$f_x(x,y) = e^x \cos y$, $\quad f_x(0,0) = 1$

$f_y(x,y) = -e^x \sin y$, $\quad f_y(0,0) = 0$.

$\therefore f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b) + \ldots]$

At (0,0), $f(x,y) = f(0,0) + (x)f_x(0,0) + (y)f_y(0,0) + \ldots$

$\mathbf{e^x \cos y} = 1 + x + \ldots$

17. **State the conditions for maxima and minima of $f(x,y)$.**

Solution:

If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$ then

(i) $f(x,y)$ is maximum value at (a,b) if $AC-B^2>0$ and $A<0$

(ii) $f(x,y)$ is minimum value at (a,b) if $AC-B^2>0$ and $A>0$

18. Find the maxima and minima of $f(x, y) = x^2 + y^2 + 6x + 12$

Solution:

$f_x = 2x + 6 = 0 \Rightarrow x = -3$; $f_y = 2y = 0 \Rightarrow y = 0$.

The stationary point is (-3,0).

$A = f_{xx} = 2$; $B = f_{xy} = 0$; $C = f_{yy} = 2$,

$AC - B^2 = 4 > 0$ and $A > 0$.

$\therefore f$ is minimum at (–3,0) and the minimum value is

$f(-3,0) = (-3)^2 + 0^2 + 6(-3) + 12 = 21 - 18 = 3$.

19. Find the possible extreme point of $f(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$

Solution:

$$f(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$$

$$\frac{\partial f}{\partial x} = 2x - \frac{2}{x^2}; \quad \frac{\partial f}{\partial x} = 0 \Rightarrow x = 1$$

$$\frac{\partial f}{\partial y} = 2y - \frac{2}{y^2}; \quad \frac{\partial f}{\partial y} = 0 \Rightarrow y = 1$$

$\therefore$ The possible extreme point is $(1,1)$.

20. Find the stationary points of $f(x,y) = x^3 - 3y + y^3 - 12x + 20$

Solution:

$f(x,y) = 3x^2 - 12 = 0$, , $f(x,y) = 3y^2 - 3 = 0$.

$3x^2 - 12 = 0$		$3y^2 - 3 = 0$
$\Rightarrow x^2 - 4 = 0$	,	$\Rightarrow y^2 - 1 = 0$
$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$		$\Rightarrow y = \pm 1$

$\therefore$ The stationary point is (2,1),(2,-1),(-2,1)(-2,-1).

UNIT IV - INTEGRAL CALCULUS

PART-A

Integral calculus is a branch of calculus that deals with the concept of integration. Integration is essentially the reverse process of differentiation, and it is used to calculate quantities such as areas, volumes, and accumulated quantities like distance, velocity, and more. It has many practical applications in science, engineering, economics and various other fields.

Definite and Indefinite Integrals: There are two main types of integrals - definite and indefinite integrals. An indefinite integral, denoted by ∫f(x)dx, represents a family of functions that have the same derivative (up to a constant). It is often written as F(x) + C, where F(x) is an antiderivative of f(x) (a function whose derivative is f(x)), and C is the constant of integration.

Application: Area and Volume Calculations: One of the fundamental applications of integration is in calculating areas under curves and volumes of irregularly shaped objects. For example, you can use integration to find the area under a curve in a graph or the volume of a three-dimensional object.

1 **State the fundamental theorem of calculus.**

Solution: Suppose f is continuous on $[a, b]$

- If $g(x) = \int_a^x f(t)dt, \quad then\ g'(x) = f(x)$
- $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivatives of f, that is $F' = f$.

2 **State the properties of Integrals.**

Solution:

(i) If k is any constant, then $\int k\ f(x)dx = k\int f(x)dx$

(ii) If $f(x)$ and $g(x)$ are any two functions in x then $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

3 **State the properties of definite integrals.**

Solution:

(i) If $f(x)$ is odd function of x then $\int_{-a}^{a} f(x)\,dx = 0$

(ii) If $f(x)$ is even function of x then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

4 **What is wrong with the equation** $\int_{-1}^{2} \frac{4}{x^3}\,dx = \left[\frac{-2}{x^2}\right]_{-1}^{2} = \frac{3}{2}$?

Solution: The given function $\frac{4}{x^3}$ is not continuous at $x = 0$. Hence the given integration does not exist. [By the fundamental theorem of calculus] i. e. the function f(x) has infinite discontinuity at $x = 0$. Hence $\int_{-1}^{2} \frac{4}{x^3}\,dx$ does not exist.

5 **If f is continuous and** $\int_{0}^{4} f(x)dx = 10$**, find** $\int_{0}^{2} f(2x)dx$**.**

Solution: Since f is continuous, let $2x = t$, differentiating $2dx = dt$ or $dx = dt/2$. When $x = 2$, then $t = 4$.

$$\therefore \int_{0}^{2} f(2x)dx = \int_{0}^{4} f(t)\frac{dt}{2} = \frac{1}{2}\int_{0}^{4} f(t)dt = \frac{10}{2} = 5$$

6 **Evaluate** $\int_{4}^{\infty} \frac{1}{\sqrt{x}}\,dx$ **and determine whether it is convergent or divergent.**]

Solution:

$$\int_{4}^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{x\to\infty}\int_{4}^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{x\to\infty}\left[\frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right]_{4}^{x} = \lim_{x\to\infty}\left[2x^{\frac{1}{2}}\right]_{4}^{x} = 2\lim_{x\to\infty}\left[\sqrt{x}-\sqrt{4}\right] = 2\left[\infty-\sqrt{4}\right] = \infty$$

The limit does not exist. Hence it is divergent.

7 **Examine whether** $\int_{-\infty}^{0} 2^{r}\,dr$ **is convergent or divergent?**

Solution:

$$\int_{-\infty}^{0} 2^{r}\,dr = \underset{t\to-\infty}{Lt}\int_{t}^{0} 2^{r}\,dr = \underset{t\to-\infty}{Lt}\left[\frac{2^{r}}{\log 2}\right]_{t}^{0} = \frac{1}{\log 2} - \underset{t\to-\infty}{Lt}\left(\frac{2^{t}}{\log 2}\right) = \frac{1}{\log 2} - 0 = \frac{1}{\log 2} = \text{finite}$$

So the integral is convergent.

8 **Evaluate** $\int_{3}^{\infty} \frac{dx}{(x-2)^{\frac{3}{2}}}$ **and determine whether it is convergent or divergent. [Apr/May 2019]**

Solution:

$$\text{Let } I = \int_{3}^{\infty} \frac{dx}{(x-2)^{\frac{3}{2}}} = \lim_{b\to\infty}\int_{3}^{b} \frac{dx}{(x-2)^{\frac{3}{2}}} = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{x-2}}\right)_{3}^{b} = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b-2}} + \frac{-2}{\sqrt{3-2}}\right) = 2$$

Hence the given integral is convergent.

9 **Find** $\int e^{x\log 2} e^x \, dx$

Solution: $\int e^{x\log 2} e^x \, dx = \int e^{\log 2^x} e^x \, dx = \int 2^x e^x \, dx = \int (2e)^x \, dx = \dfrac{(2e)^x}{\log 2e} + c$

10 **Find** $\int \dfrac{\cos x}{\sqrt{\sin x}} dx$ **by substitution method.**

Solution:

Put $\mathrm{u} = \sin \mathrm{x}$ then $\cos \mathrm{x}\, \mathrm{dx} = \mathrm{du} \therefore \mathrm{I} = \int \dfrac{\mathrm{du}}{\sqrt{\mathrm{u}}} = 2\sqrt{\mathrm{u}} + \mathrm{c} = 2\sqrt{\sin \mathrm{x}} + \mathrm{c}$

11 ***Evaluate*** $\int \dfrac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Solution:

Put $e^x + e^{-x} = t$

$\left(e^x - e^{-x}\right) dx = dt$

$\int \dfrac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \dfrac{dt}{t} = \log t = \log\left(e^x + e^{-x}\right) + c$

12 ***Evaluate*** $\int \tan^{-1} x \, dx$

Solution: *Take* $u = \tan^{-1} x, \quad dv = dx$

$$du = \frac{1}{1+x^2} dx, \quad v = x$$

$$\int \tan^{-1} x \, dx = uv - \int v du$$

$= x \tan^{-1} x - \int x \dfrac{dx}{1+x^2}$

$= x \tan^{-1} x - \dfrac{1}{2}\int \dfrac{2x}{1+x^2} \qquad \because \int \dfrac{f'(x)}{f(x)} dx = \log f(x)$

$= x \tan^{-1} x - \dfrac{1}{2}\log\left(1+x^2\right) + c$

13 **Evaluate** $\int \dfrac{dx}{\left(1+x^2\right)^2}$

Solution: Put $x = \tan\theta, \; dx = \sec^2\theta \, d\theta$, $\int \dfrac{dx}{\left(1+x^2\right)^2} = \int \dfrac{\sec^2\theta \, d\theta}{\left(1+\tan^2\theta\right)^2}$

$= \int \cos^2\theta \, d\theta = \int \left(\dfrac{1+\cos 2\theta}{2}\right) d\theta$

$= \dfrac{1}{2}\int (1+\cos 2\theta) d\theta = \dfrac{1}{2}\left(\theta + \dfrac{\sin 2\theta}{2}\right) = \left(\dfrac{\theta}{2} + \dfrac{\sin 2\theta}{4}\right) + c$

14 Evaluate $\int_1^2\left(-3x^{\frac{1}{2}}+\frac{1}{x^2}\right)dx$

Solution:

$$\int_1^2\left(-3x^{\frac{1}{2}}+\frac{1}{x^2}\right)dx=-3\int_1^2 x^{\frac{1}{2}}dx+\int_1^2\frac{1}{x^2}dx=\left[-2x^{\frac{3}{2}}-\frac{1}{x}\right]_1^2$$

$$=\left[\left(-2\left(2^{\frac{3}{2}}\right)-\frac{1}{2}\right)-\left(-2\left(1^{\frac{3}{2}}\right)-1\right)\right]$$

$$=\left[\left(-2\left(2^{\frac{3}{2}}\right)-\frac{1}{2}\right)+3\right]=-4\sqrt{2}+\frac{5}{2}$$

15 Evaluate $\int \tan^3 x\,dx$

Solution:

$$\int \tan^3 x\,dx=\int \tan x\tan^2 x\,dx=\int \tan x\left(\sec^2 x-1\right)dx$$

$$=\int\left(\tan x\sec^2 x-\tan x\right)dx$$

put $u=\tan x \Rightarrow du=\sec^2 x\ dx$ in the first integral

$$=\int u\,du-\int \tan x\,dx$$

$$=\frac{u^2}{2}-\log\left(\sec x\right)+c\ =\ \frac{\tan^2 x}{2}-\log\left(\sec x\right)+c$$

16 Evaluate $\int\frac{dx}{1+\cos x}$ **by decomposition method.**

Solution: Multiply and divide by conjugate factor of the denominator term

$$\int\left(\frac{1}{1+\cos x}\right)\left(\frac{1-\cos x}{1-\cos x}\right)dx=\int\left(\frac{1-\cos x}{1-\cos^2 x}\right)dx$$

$$=\int\left(\frac{1-\cos x}{\sin^2 x}\right)dx=\int\left[\frac{1}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right]dx=\int\left[\operatorname{cosec}^2 x-\operatorname{cosec} x\cot x\right]dx$$

$$=\int \operatorname{cosec}^2 x\,dx-\int \operatorname{cosec} x\cot x\,dx=-\cot x-\left(-\operatorname{cosec} x\right)+c$$

$$\therefore\int\frac{1}{1+\cos x}dx=\operatorname{cosec} x-\cot x+c$$

17 Find the Integral of $x\sin x$ **using integration by parts.**

Solution:

$$\int u\,dv=uv-\int v\,du\quad;\text{put }\ u=x\ \text{and}\ dv=\sin x\,dx, du=dx, v=\int dv=\int \sin x\,dx=-\cos x$$

$$\int x\sin x\ dx=x\left(-\cos x\right)-\int\left(-\cos x\right)dx=-x\cos x+\sin x+c=\sin x-x\cos x+c$$

18 Evaluate $\int x\log x\,dx$ **using integration by parts.**

Solution: $\int u\,dv=uv-\int v\,du\quad;\text{put }\ u=\log x\ \ and\ \ dv=x\,dx, du=\frac{dx}{x}, v=\int dv=\int x\,dx=\frac{x^2}{2}$

$$\int x\log x\,dx=\frac{x^2}{2}\log x-\int\frac{x^2}{2}dx=\frac{x^2}{2}\log x+\frac{x^3}{6}+c$$

19 **Evaluate** $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^5 x\, dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n}\frac{n-3}{m+n-2}\frac{n-5}{m+n-4}\cdots\cdots\frac{1}{n+2}$$

here $m=6, n=5$

$$= \frac{4}{11}\,\frac{2}{9}\,\frac{1}{7} = \frac{8}{693}$$

20 **Evaluate** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}$

Solution:

$$Let\ I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{\sin x}{\cos x}} = \int_0^{\frac{\pi}{2}} \frac{\cos x\ dx}{\cos x+\sin x} ------(1)$$

$$\text{Since} \int_0^a f(x)dx = \int_0^a f(a-x)dx \ \Rightarrow \int_0^{\frac{\pi}{2}} \frac{\cos x\ dx}{\cos x+\sin x} = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-x\right)dx}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)} = \int_0^{\frac{\pi}{2}} \frac{\sin x\ dx}{\cos x+\sin x}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x\ dx}{\cos x+\sin x} ------(2)$$

$$(1)+(2) \Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x\ dx}{\cos x+\sin x} = \frac{\pi}{2}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} = \frac{\pi}{4}$$

UNIT – IV MULTIPLE INTEGRALS

PART – A

1. **Evaluate** $\int_1^a \int_2^b \frac{dxdy}{xy}$.

Solution: $I = \int_1^a \int_2^b \frac{dxdy}{xy} = \int_1^a \int_2^b \left(\frac{dx}{x}\right)\frac{dy}{y} = \int_1^a \frac{1}{y}[\log x]_2^b dy = \int_1^a [\log b - \log 2]\frac{1}{y}dy = \log\left(\frac{b}{2}\right)\int_1^a \frac{1}{y}dy$

$$= \log\left(\frac{b}{2}\right)[\log y]_1^a = \log\left(\frac{b}{2}\right)[\log a - \log 1] = \log\left(\frac{b}{2}\right)\log a. \qquad \text{[Since } \log 1 = 0\text{]}$$

2. **Evaluate** $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy$.

Solution:

$$I = \int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy = \int_1^{\ln 8}\int_0^{\ln y} e^x e^y dxdy = \int_1^{\ln 8} e^y \left[e^x\right]_0^{\ln y} dy$$

$$= \int_1^{\ln 8} e^y \left(e^{\ln y} - e^0\right)dy = \int_1^{\ln 8} e^y (y-1)\,dy = \left[e^y(y-1) - e^y\right]_1^{\ln 8} \qquad \because \int udv = uv - u'v_1 + \ldots$$

$$= \left[e^{\ln 8}(\ln 8 - 1) - e^{\ln 8}\right] - \left[0 - e^1\right] = e^{\ln 8}(\ln 8 - 1 - 1) + e = 8(\ln 8 - 1 - 1) + e = 8\ln 8 - 16 + e.$$

3. **Evaluate the double integral** $\int_0^1 \int_0^{x^2} \left(x^2 + y^2\right)dxdy$

Solution: The given order of integration is not correct. So Rewrite the integration order as

$$\int_0^1 \int_0^{x^2} (x^2 + y^2)dxdy = \int_0^1 \int_0^{x^2} (x^2 + y^2)dydx = \int_0^1 \left[\int_0^{x^2} (x^2 + y^2)dy\right]dx$$

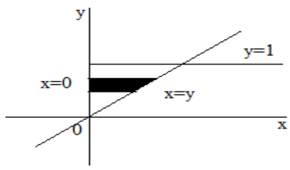
$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{x^2} dx = \int_0^1 \left(x^2.x^2 + \frac{\left(x^2\right)^3}{3}\right)dx = \int_0^1 \left(x^4 + \frac{x^6}{3}\right)dx$$

$$= \left[\frac{x^5}{5} + \frac{x^7}{21}\right]_0^1 = \left[\frac{1}{5} + \frac{1}{7.3}\right] = \left[\frac{1}{5} + \frac{1}{21}\right] = \frac{26}{105}$$

4 **Find the area bounded by the lines x = 0, y = 1, y = x using double integration.**

Solution:

$$Area = \iint_R dxdy = \int_0^1 \int_0^y dxdy = \int_0^1 [x]_0^y dy = \int_0^1 ydy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}.$$



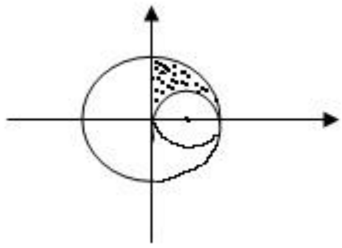
5. **Shade the region of integration in** $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dxdy.$

Solution: The given order of the integration is not correct. So Rewrite order $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dydx.$

$y=\sqrt{ax-x^2} \Rightarrow x^2+y^2-ax=0$

$\Rightarrow x^2-ax+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+y^2=0$

$\Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

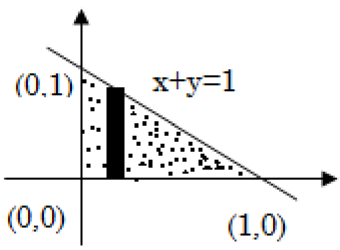


which is a circle with centre at ($a/2$, 0) and radius $a/2$

y = $\sqrt{a^2-x^2} \Rightarrow x^2+y^2=a^2$ which is a circle with centre at (0,0) and radius a

6. **Evaluate $\iint_R (x^2+y^2)\,dy\,dx$ over the region R for which x, y$\geq$ 0, x + y$\leq$1.**

Solution: The region of integration is the triangle bounded by the lines x = 0, y = 0 and x + y = 1. Limits of y : 0 to 1-x
Limits of X: 0 to 1.



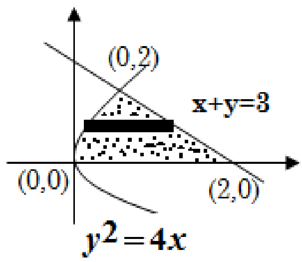
$$\iint_R (x^2+y^2)\,dy\,dx = \int_0^1\int_0^{1-x}(x^2+y^2)\,dydx = \int_0^1\left[x^2y+\frac{y^3}{3}\right]_0^{1-x} dx$$

$$=\int_0^1\left[x^2(1-x)+\frac{(1-x)^3}{3}\right] dx = \int_0^1\left[x^2-x^3+\frac{(1-x)^3}{3}\right] dx$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{(1-x)^4}{(3)(4)(-1)}\right]_0^1 = \left(\frac{1}{3}-\frac{1}{4}+0\right)-\left(-\frac{1}{12}\right)=\frac{1}{3}-\frac{1}{4}+\frac{1}{12}=\frac{1}{6}$$

7. **Compute the area enclosed by y^2 = 4x, x + y = 3 and y = 0.**

Solution: Area A = $\iint_R dx\,dy = \int_{y=0}^{2}\int_{x=y^2/4}^{3-y} dx\,dy = \int_{y=0}^{2}[x]_{y^2/4}^{3-y}\,dy$

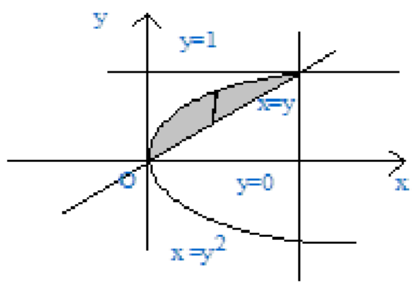


$$=\int_{y=0}^{2}\left[3-y-\frac{y^2}{4}\right]dy = \left[3y-\frac{y^2}{2}-\frac{y^3}{12}\right]_0^2$$

$$= 6-\frac{4}{2}-\frac{8}{12} = 6-2-\frac{8}{12} = 4-\frac{2}{3} = \frac{10}{3}$$

8. **Change the order of integration in $\int_0^1\int_{y^2}^{y} f(x,y)\,dxdy$.**

Solution:



Given, I = $\int_0^1\int_{y^2}^{y} f(x,y)\,dxdy$..

Limits of X: y^2 to y and Limits of Y: 0 to 1
After changing order of integration

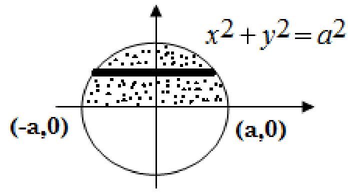
$$I=\int_0^1\int_x^{\sqrt{x}} f(x,y)\,dydx$$

9. **Change the order of integration in $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy$.**

Solution:

$$I = \int_{x=-a}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx \text{ (Correct Form)}$$

$$I = \int_{y=0}^{y=a} \int_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} (x^2 + y^2)\; dy \; dx$$

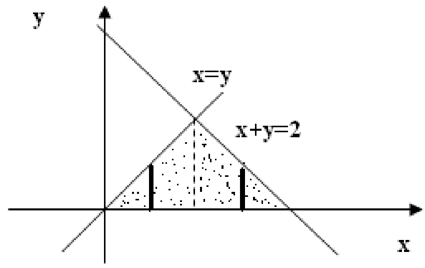


10. **Change the order of integration in $\int_0^1 \int_y^{2-y}$ xy dx dy** .

Solution: Given, I = $\int_0^1 \int_y^{2-y} xy\ dx\ dy$

x limits : y to 2-y; y limits : 0 to 1

After changing order of integration



$$I = \int_0^1 \int_0^x xy\ dy\ dx + \int_1^2 \int_0^{2-x} xy\ dy\ dx$$

11. **Evaluate $\int_0^{\pi} \int_0^5 r\sin^2\theta\, dr\, d\theta$.**

Solution: I = $\int_0^{\pi} \sin^2\theta \left[\int_0^5 r\,dr\right] d\theta = \int_0^{\pi} \sin^2\theta \left[\frac{r^2}{2}\right]_0^5 d\theta$ =

$$\frac{25}{2}\int_0^{\pi} \sin^2\theta\ d\theta = \frac{25}{2}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\ d\theta$$

$$= \frac{25}{4}\int_0^{\pi} [1-\cos 2\theta]\ d\theta = \left(\frac{25}{4}\right)\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \left(\frac{25}{4}\right)\left[\left\{\pi - \frac{\sin 2\pi}{2}\right\} - 0\right] = \frac{25\pi}{4}$$

12. **Evaluate $\int_0^{\pi/2} \int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$.**

Solution: I = $\int_0^{\pi/2} \int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2} = \int_0^{\pi/2} \frac{1}{2}\left(\int_0^{\infty} \frac{d\left(r^2+a^2\right)}{(r^2+a^2)^2}\right) d\theta$

$$= \int_0^{\pi/2} \frac{1}{2}\left[\frac{-1}{r^2+a^2}\right]_0^{\infty} d\theta \qquad \because \int [f(x)]^n\, d(f(x)) = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[0 + \frac{1}{a^2}\right] d\theta = \frac{1}{2}\left(\frac{1}{a^2}\right) [\theta]_0^{\pi/2} = \frac{1}{2}\left(\frac{1}{a^2}\right)\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4a^2}$$

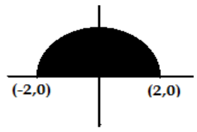
13. **Evaluate** $\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$.

Solution:

$$I=\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\cos\theta}=\frac{1}{2}\int_0^{\pi}\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{\pi}\frac{1+\cos 2\theta}{2}d\theta=\frac{1}{4}\int_0^{\pi}(1+\cos 2\theta)\,d\theta$$

$$=\frac{1}{4}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi}=\frac{1}{4}(\pi)=\frac{\pi}{4}$$

14. **Evaluate** $\iint_R dxdy$ **where** R **the shaded region in the figure is.**

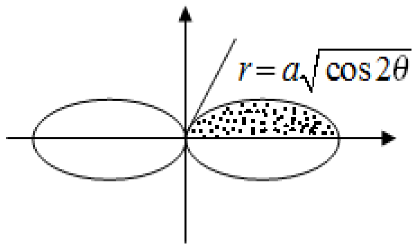


Solution: $\iint_R dxdy = \text{Area of the shaded region}$

The shaded region is the semicircle with radius 2.

Area of the shaded portion $=\frac{\pi r^2}{2}=\frac{\pi 2^2}{2}=2\pi$ *square units*

15. **Compute the entire area bounded by** $r^2 = a^2\cos 2\theta$.



Solution: Area A = $\iint_R r\,dr\,d\theta = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$

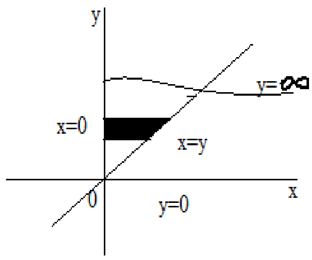
$$=4\int_{\theta=0}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}d\theta$$

$$=4\int_0^{\pi/4}\left[\frac{a^2\cos 2\theta}{2}\right]d\theta$$

$$=2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}=2a^2\left[\frac{\sin 2\frac{\pi}{4}}{2}-0\right]=a^2\sin\frac{\pi}{2}=a^2$$

16. **Transform the integration** $\int_0^{\infty}\int_0^{y} dx\,dy$ **into polar coordinates.**

Solution: Let x = r cosθ and y = r sinθ , dxdy = r drdθ



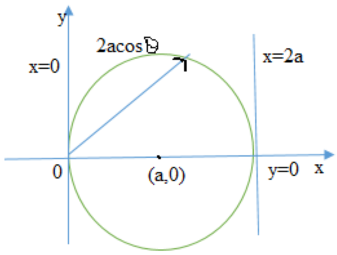
$$\int_0^{\infty}\int_0^{y} dx\,dy = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{\infty} r\,dr\,d\theta$$

17. **Transform the integration** $\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy$ **to polar co-ordinates**

Solution:

Let x = r cosθ and y = r sinθ, dxdy = r drdθ

$$\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$



18. **Express the volume of the region bounded by $x\ge 0,\ y\ge 0,\ z\ge 0,\ x^2+y^2+z^2\le 1$ as a triple integral.**

Solution: Here z varies from 0 to $\sqrt{1-x^2-y^2}$, y varies from 0 to $\sqrt{1-x^2}$, x varies from 0 to 1

$$\therefore Volume\ I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

19. **Evaluate** $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dxdydz$

Solution: I = $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dxdydz = \int_0^1\int_0^1\left[e^{1+y+z}-e^{y+z}\right]dydz$

$$= \int_0^1\left(e^{z+2}-2e^{z+1}+e^z\right)dz = e^3-3e^2+3e-1=(e-1)^3$$

20. **Evaluate** $\int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dx\,dy\,dz.$

Solution: I = $\int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dz\,dy\,dx = \int_0^4\int_0^x\left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}}dydx = \frac{1}{2}\int_0^4\int_0^x(x+y)dydx$

$$= \frac{1}{2}\int_0^4\left(xy+\frac{y^2}{2}\right)_0^x dx = \frac{1}{2}\int_0^4\left(x^2+\frac{x^2}{2}\right)dx = \frac{3}{4}\int_0^4 x^2dx = \frac{3}{4}\left(\frac{x^3}{3}\right)_0^4 = 16$$

DMI College of Engineering -Palanchur

MA3151 – MATRICES AND CALCULUS

Unit-1 - MATRICES

Matrices – Introduction

Matrices are fundamental mathematical objects used to represent and manipulate data in various fields, including mathematics, physics, computer science, engineering, and more. They are an essential part of linear algebra, which is the branch of mathematics that deals with vector spaces and linear relationships.

Definition: **A matrix is a two-dimensional array of numbers, symbols, or expressions, organized into rows and columns. Each entry in a matrix is called an element or a coefficient (Or) arrange the elements in the rectangular patterns.**

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [A] with elements a_{ij}

$$\begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{in} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

Order of the matrix or Size: The order or size of a matrix is determined by the number of rows and columns. For example, a matrix with 3 rows and 2 columns is called a "3x2" matrix.

Types of Matrices: There are various types of matrices based on their properties and characteristics. Some common types include:

Row Matrix: **A matrix with a single row and multiple columns.**

Column Matrix: **A matrix with a single column and multiple rows.**

Square Matrix: A matrix with an equal number of rows and columns (e.g., 2x2, 3x3, etc.).

Scalar Matrix: A square matrix with all diagonal elements equal and all other elements equal to zero.

Identity Matrix: **A special scalar matrix where all diagonal elements are 1 and all other elements are 0.**

Zero Matrix: **A matrix where all elements are zero.**

Row Vector: **A matrix with one row and multiple columns.**

Column Vector: **A matrix with one column and multiple rows.**

Symmetric Matrix: **A symmetric matrix is a square matrix that is equal to its transpose. In other words, A = AT, where A is the matrix and AT is its transpose.**
Skew Symmetric Matrix: **A symmetric matrix is a square matrix that is equal to its transpose. In other words, A = -AT, where A is the matrix and AT is its transpose.**

Lower triangular matrix
A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix}$$

Upper triangular matrix
A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Eigen values and Eigenvectors:

Linear equations Ax= b come from steady state problems. Eigen values have their greatest importance in dynamic problems. This chapter enters a new part of linear algebra, based on Ax= λ x. All matrices in this chapter are square. Where λ is a eigen values and x is a eigen vectors. The Characteristic equation is det$|A - \lambda I| = 0$ involves only λ not in x. Its roots are called eigen values and $(A - \lambda I)X = 0$ corresponding eigenvectors. (Or) To find the Characteristic equation ,

For the 2X2 matrix: If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

The Char. equation $\lambda^2 - s_1 \lambda - s_2 = 0$

Where s_1 = Sum of the main diagonal elements = $a_{11} + a_{22}$

s_2 = Determinant of A = $|A| = (a_{11}a_{22} - a_{21}a_{12})$

For the 3X3 matrix: If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

The Char. equation $\lambda^3 - s_1 \lambda^2 + s_2 \lambda - s_3 = 0$

Where s_1 = Sum of the main diagonal elements = $(a_{11} + a_{22} + a_{33})$

s_2 = Sum of the minor of main diagonal elements

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

s_3 = Determinant of A = $|A|$

******Cayley – Hamilton Theorem:**

Every square matrix satisfies its own characteristic equation.

To uses:

(i) To calculate positive integral power A

(ii) To Find the inverse of a square matrix A

1. **Verify Cayley – Hamilton theorem (OR) Prove that Every square matrix satisfies its own characteristic equation, if** $A = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{pmatrix}$

Soln:

The characteristic equation is $\lambda^3 - s_1 \lambda^2 + s_2 \lambda - s_3 = 0$

s_1 = Sum of the main diagonal elements = (1 + 2 + 2) = $\mathbf{5}$

s_2 = Sum of the minor of main diagonal elements

$= \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}$

= (4 – 0) + (2 – 0) + (2 – 0)

= 4 + 2 + 2 = 8

$\boldsymbol{s_2 = 8}$

$s_3 = |A| = 1\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}$

= (4 – 0) + (0) + (2 – 0)

$\boldsymbol{s_3 = 4}$

$\lambda^3 - \mathbf{5}\lambda^2 + \mathbf{8}\lambda - \mathbf{4} = 0$

By CH- theorem,

Every square matrix satisfies its own characteristic equation. Put $\lambda = A$

$\therefore\ A^3 - 5A^2 + 8A - 4I = 0$ .... (1)

LHS: $A^3 - 5A^2 + 8A - 4I$

Now $A^2 = A.A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+0+0 & 0+0+0 & 1+0+2 \\ 2+4+0 & 0+4+0 & 2+8+8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 3 \\ 6 & 4 & 18 \\ 0 & 0 & 4 \end{pmatrix}$$

$$.A^3 = A^2A = \begin{pmatrix} 1 & 0 & 3 \\ 6 & 4 & 18 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 0+0+0 & 1+0+6 \\ 6+8+0 & 0+8+0 & 6+16+36 \\ 0+0+0 & 0+0+0 & 0+0+8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 7 \\ 14 & 8 & 58 \\ 0 & 0 & 8 \end{pmatrix}$$

LHS: $A^3 - 5A^2 + 8A - 4I$

$$= \begin{pmatrix} 1 & 0 & 7 \\ 14 & 8 & 58 \\ 0 & 0 & 8 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 3 \\ 6 & 4 & 18 \\ 0 & 0 & 4 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-5+8-4 & 0+0+0 & 7-15+8+0 \\ 14-30+16+0 & 8-20+16-4 & 58-90+32+0 \\ 0+0+0 & 0+0+0 & 8-20+16-4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 = \text{RHS}$$

Hence cayley – Hamilton theorem verified.

2. **Verify Cayley – Hamilton theorem for the matrix** $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \end{pmatrix}$

Soln:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = Sum of the main diagonal elements = (1 + 2 + 1) = $\mathbf{4}$

s_2 = Sum of the minor of main diagonal elements

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

= (2 – 6) + (1 – 7) + (2 – 12)

= −4 – 6 – 10 = −20

$\boldsymbol{s_2 = -20}$

$$s_3 = |A| = 1\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix}$$

= 1(2 – 6) – 3(4 – 3) + 7(8 – 2)

= −4 – 3 + 42 = 35

$\boldsymbol{s_3} = \mathbf{35}$

$\lambda^3 - \mathbf{4}\lambda^2 + 20\lambda - \mathbf{35} = 0$

By CH- theorem,

every square matrix satisfies its own characteristic equation. Put $\lambda = A$

$\therefore\ A^3 - 4A^2 + 20A - 35I = 0$ .... (1)

LHS: $A^3 - 4A^2 + 20A - 35I$

Now $A^2 = A.A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$.A^3 = A^2 A = \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{pmatrix}$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix}$$

LHS: $A^3 - 4A^2 + 20A - 35I$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} - 4\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 20\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - 35\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 135-80-20-35 & 152-92-60+0 & 232-92-140+0 \\ 140-60-80+0 & 163-88-40-35 & 208-148-60+0 \\ 60-40-20+0 & 76-36-40+0 & 111-56-20-35 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 = \text{RHS}$$

Hence cayley – Hamilton theorem verified.

3. Using Cayley – Hamilton theorem, find A^{-1} for the matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

Soln:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = Sum of the main diagonal elements = 2 + 2 + 2 = $\mathbf{6}$

s_2 = Sum of the minor of main diagonal elements

$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= (4 - 1) + (4 - 2) + (4 - 1)$

$= 3 + 2 + 3 = 8$

$\boldsymbol{s_2 = 8}$

$s_3 = |A| = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$

$= 2(4 - 1) + 1(-2 + 1) + 2(1 - 2)$

$= 6 - 1 - 2 = 3$

$\boldsymbol{s_3 = 3}$

$\lambda^3 - \mathbf{6}\lambda^2 + 8\lambda - \mathbf{3} = 0$

By CH- theorem,

every square matrix satisfies its own characteristic equation. Put $\lambda = A$

$\therefore\ A^3 - 6A^2 + 8A - 3I = 0$ .... (1)

To find A^{-1} $\therefore A^3 - 6A^2 + 8A - 3I = 0$ ----①

Pre multiply by A^{-1} in 1 ①

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 6A + 8I$$

Now $A^2 = A.A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$

$\therefore\ A^{-1} = \frac{1}{3}[A^2 - 6A + 8I] = \frac{1}{3}\left[\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right]$

$= \frac{1}{3}\begin{bmatrix} 7 - 12 + 8 & -6 + 6 + 0 & 9 - 12 + 0 \\ -5 + 6 + 0 & 6 - 12 + 8 & -6 + 6 + 0 \\ 5 - 6 + 0 & -5 + 6 + 0 & 7 - 12 + 8 \end{bmatrix}$

$\therefore\ A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$

Eigen value and Eigen Vector:

1. Find the eigen value and eigen vector for the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Soln:

Let $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

The char eqn is $\lambda^3 - s_1 \lambda^2 + s_2 \lambda - s_3 = 0$----------(1)

$S_1 = sum\ of\ the\ main\ diagonal = 18$

$S_2 = sum\ of\ the\ minor\ of\ main\ diagonal = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 45$, $\quad S_3 = |A| = 0$

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0, \lambda = 3, \lambda = 15$

The eigen values are 0, 3, 5.

Eigen vectors

$$(A - \lambda I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{array}{r} (8-\lambda)x_1 - 6x_2 + 2x_3 = 0 \\ 6x_1 + (7-\lambda)x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + (3-\lambda)x_3 = 0 \end{array}\right\} \text{--------(A)}$$

Put $\lambda = 0$ in (A)

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$6x_1 + 7x_2 - 4x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

When $\lambda = 3$ in

$$x_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

When $\lambda = 15$ in

$$x_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

2. **Find the Eigen value and eigen vector for** $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Soln:

Let $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

The char eqn is $A^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$

$S_1 = 0$ $S_2 = -3$ $S_3 = -2$

$$\therefore \lambda^3 - 0\lambda^2 - 3\lambda + 2 = 0$$

$eigenvalues\ are$ $-2, 1, 1$

To find Eigen values $(A - \lambda I)x = 0$

$$\begin{pmatrix} 0-\lambda & 1 & -1 \\ 1 & 0-\lambda & 1 \\ -1 & 1 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\lambda x_1 + x_2 - x_3 = 0$$

$$x_1 - \lambda x_2 + x_3 = 0$$

$$-x_1 + x_2 - \lambda x_3 = 0$$

When $\lambda = -2$ in ----①

$2x_1 + x_2 - x_3 = 0$----①

$x_1 + 2x_2 + x_3 = 0$----②

$-x_1 + x_2 + 2x_3 = 0$--------③

Taking,

1&2^{nd} eqns

$$\frac{x_1}{3} = \frac{-x_2}{3} = \frac{x_3}{3}$$

$x_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

When $\lambda = 1$ in ----①

$x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Since A is a symmetric matrix so Given Eigen vectors are orthogonal.

$let\ x_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the 3rd eigen vector

$x_1^T x_3 = 0a - b + c = 0$----①

$x_1^T x_3 = 00a - b + c = 0$----②

$$\frac{a}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & -0 \\ 0 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{a}{-2} = \frac{-b}{1} = \frac{c}{1}$$

****16 Marks Reduce the Quadratic form for** $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

$$Q = \begin{bmatrix} coeff\ x^2 & \frac{1}{2} coeff\ xy & \frac{1}{2} coeff\ xz \\ \frac{1}{2} coeff\ yx & coeff\ y^2 & \frac{1}{2} coeff\ yz \\ \frac{1}{2} coeff\ zx & \frac{1}{2} coeff\ zy & coeff\ z^2 \end{bmatrix}$$

1. Reduce the Quadratic form to canonilal form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3$ by an orthogonal transformation. Also find the rank, index and signature.

Soln:

Let $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

The char eqn is $\lambda^3 - s_1 \lambda^2 + s_2 \lambda - s_3 = 0$----------(1)

$S_1 = sum\ of\ the\ main\ diagonal = 18$

$S_2 = sum\ of\ the\ minor\ of\ main\ diagonal = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 45$, $\quad S_3 = |A| = 0$

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0, \lambda = 3, \lambda = 15$

The eigen values are 0, 3, 5.

Eigen vectors

$$(A - \lambda I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$(8-\lambda)x_1 - 6x_2 + 2x_3 = 0$$
$$6x_1 + (7-\lambda)x_2 - 4x_3 = 0 \quad \text{--------(A)}$$
$$2x_1 - 4x_2 + (3-\lambda)x_3 = 0$$

Put $\lambda = 0$ in (A)

$$8x_1 - 6x_2 + 2x_3 = 0$$
$$6x_1 + 7x_2 - 4x_3 = 0$$
$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$
$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

When $\lambda = 3$ in

$$x_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

When $\lambda = 15$ in

$$x_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

P=Modal matrix $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

The normalised matrix is $N = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$

$D = N^T A N$

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

Canonical form = $y^{-1} D Y = 0y_1^2 + 3y_2^2 + 15y_3^2$

$$r = Rank = 2$$

$$P = positive\ term = Index = 2$$

Signature $S = 2P - r = 2(2) - 2 = 2$

Nature of Q. F :Semi Positive definite.

2. Reduce the quadratic form $2x_1x_2+2x_2x_3-2x_3x_1$ to canonical form by an orthogonal transformation. Also find the rank index and signature of the Q.F.

Soln:

Reduce the matrix form is $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

The char eqn is$A^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$

$S_1 = 0 \quad S_2 = -3 \qquad S_3 = -2$

$$\therefore \lambda^3 - 0 \lambda^2 - 3 \lambda + 2 = 0$$

$eigenvalues\ are$ $-2, 1, 1$

To find Eigen values$(A - \lambda I)x = 0$

$$\begin{pmatrix} 0-\lambda & 1 & -1 \\ 1 & 0-\lambda & 1 \\ -1 & 1 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\lambda x_1 + x_2 - x_3 = 0$$

$$x_1 - \lambda x_2 + x_3 = 0$$

$$-x_1 + x_2 - \lambda x_3 = 0$$

When $\lambda = -2$ in ----①

$2x_1 + x_2 - x_3 = 0$----①

$x_1 + 2x_2 + x_3 = 0$----②

$-x_1 + x_2 + 2x_3 = 0$--------③

Taking,

1&2^{nd} eqns

$$\frac{x_1}{3} = \frac{-x_2}{3} = \frac{x_3}{3} x_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

When $\lambda = 1$ in ----①

$x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$let x_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$be the 3rd eigen vector

$x_1^T x_3 = 0 a - b + c = 0$----①

$x_1^T x_3 = 00a - b + c = 0$ ----②

$$\frac{a}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & -0 \\ 0 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{a}{-2} = \frac{-b}{1} = \frac{c}{1}$$

Mode matrix

$$M = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} x_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$D = N_A^T N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To find canonical form

$$C.F = Y^T DY = (y_1 y_2 y_3) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -2y_1^2 y_2^2 y_3^2 rank = 3$$

$$P = Index = 2$$

Signature $2(0) - r = 2(2) - 3 = 1$

Nature of the Q. F : Indefinite.

3. Reduce the Quadratic form $x^2 + 5y^2 + z^2 + 2xy + 2yz - 6zx$ to canonical form by an orthogonal transformation also find the rank index and signature of the Q.F.

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The char eqn is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$

$S_1 = 7$, $S_2 = 0$, $S_3 = 36$

$\therefore \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

$\lambda = -2, \lambda^2 - 9\lambda + 18 = 0$

$(\lambda-3)(\lambda-6)=0\ \lambda=3,6$

∴ The find Eigen vector

$$(A-\lambda I)x=0$$

$$\begin{pmatrix}1-\lambda & 0 & 0\\ 1 & 5-\lambda & 0\\ 3 & 1 & 1-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Case (i) When $\lambda=2$

$$\begin{pmatrix}3 & 1 & 3\\ 1 & 7 & 1\\ 3 & 1 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix}1 & 3\\ 7 & 1\end{vmatrix}}=\frac{-x_2}{\begin{vmatrix}3 & 3\\ 1 & 1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}3 & 1\\ 1 & 7\end{vmatrix}}\ x_1=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Case (ii) When $\lambda=3$

$$\begin{pmatrix}-2 & 1 & 3\\ 1 & 2 & 1\\ 3 & 1 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}}=\frac{-x_2}{\begin{vmatrix}-2 & 3\\ 1 & 1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}-2 & 1\\ 1 & 2\end{vmatrix}}\ \frac{x_1}{-5}=\frac{-x_2}{-5}=\frac{x_3}{-5}\ x_2=\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Case (iii) When $\lambda=6$

$$\begin{pmatrix}-5 & 1 & 3\\ 1 & -1 & 1\\ 3 & 1 & -5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix}1 & 3\\ -1 & 1\end{vmatrix}}=\frac{-x_2}{\begin{vmatrix}5 & 3\\ 1 & 1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}-5 & 1\\ 1 & -1\end{vmatrix}}\ \frac{x_1}{4}=\frac{-x_2}{-8}=\frac{x_3}{4}\ \therefore\ x_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$M=\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 2\\ -1 & 1 & 1\end{bmatrix}$$

$$D=N_A^T N$$

$$=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{pmatrix}$$

Canonical form

$$y_D^Y y=(y_1\ y_2\ y_3)\begin{pmatrix}-2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=-2y_1^2+3y_2^2+6y_3^2$$

Rank (r) = 3

Index (p) = 2

Signature (s) = 2(1) – r = 2 (2) – 3 = 1

The nature is indefinite

4. Reduce the Quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 + -4x_1x_2 + 4x_1x_3 - 2x_2x_3$ to canonical form. Also find rank index, signature and nature of Q.F.

Let $A = \begin{pmatrix} 6 & -2 & 2 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

The char eqn is $\lambda^3 - s_1 \lambda^2 + s_2 \lambda - s_3 = 0$

$\boxed{s_1 = 12}$, $\boxed{s_2 = 36}\boxed{s_3 = 32}$

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\boxed{\lambda = 2}$, $\lambda^2 - 10\lambda + 16 = 0$

$(A - 2)(\lambda - 8) = 0$

The eigen values are 8, 2, 2.

Eigen vector $(A - \lambda I)\, x = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(6 - \lambda)11_1 - 2x_2 + 21_3 = 0$ ________ ①

$-2x_1 + (3 - \lambda)x_2 - x_3 = 0$ ________ ②

$2x_1 - x_2 + (3 - \lambda)x_3 = 0$ ________ ③

Care (i) $\lambda = 8$

$-2x_1 - 2x_2 + 2x_3 = 0$ ________ ①

$-2x_1 - 5x_2 - x_3 = 0$ ________ ②

$x_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Care (ii) $\lambda = 2$

$-2x_1 + x_2 - x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

Answer $x_3 = 0$

$-2x_1 + x_2 = 0$

$-2x_1 = -x_2$

$\frac{x_1}{1} = \frac{x_2}{-2}$

$$x_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

Let $x_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be third eigen vector

$$(2 - 1 \quad 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \;\&(1 \quad -2 \quad 0)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$2a - b + c = 0$ __________①

$a - 2b + 0c = 0$ __________②

$$\frac{a}{\begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{a}{2} = \frac{-b}{-1} = \frac{c}{-4+1}$$

$$x_3 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 & 2 \\ -1 & -2 & 1 \\ 1 & 0 & -3 \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{6}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{16}} \\ \frac{1}{\sqrt{6}} & \frac{6}{\sqrt{5}} & \frac{-3}{\sqrt{14}} \end{pmatrix}$$

$$D = N^T AN$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Canonical form = = $y_D^T\, y$

$$= (y_1 y_2 y_3)\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$= 8y_1^2 + 2y_2^2 + 2y_3^2$

Ranu$(r) = 3, Indel = 3$

Signature $= 2P - r = 6 - 3 = 3$

Nature of Q.F = positive definite

Unit-II

Differential Calculus

16 Mark Question with Answer

1. Guess the value of the given lim function

a) $\lim_{x \to 0} \frac{x^2-1}{x-1}$

Solution

Here, $f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1$

values of x below and above 1

x / below 1	f(x) = x+1	x / above 1	f(x) = x+1
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.99	1.001	2.001
0.9999	1.999	1.0001	2.0001

f(x) tends to 2 as x tends to 1

$\Rightarrow \lim_{x \to 1} f(x) = 2$

$\Rightarrow \lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

b) $\lim_{x \to 0} \frac{\sin x}{x}$

Solution:

Given $\lim_{x \to 0} \frac{\sin x}{x}$, Here $f(x) = \frac{\sin x}{x}$

values of x below and above 0

x / below 0	f(x) = sin x/x	x / above 0	f(x) = sin x/x
-0.1	0.99883417	0.1	0.99833417
-0.01	0.9998333	0.01	0.99998333
-0.001	0.99999983	0.001	0.999999583
-0.0001	0.99999994	0.0001	0.99934665

From table $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2 Sketch the graph of the following

$f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \le x \le 1 \\ 2-x, & x \ge 1 \end{cases}$ for which $\lim_{x \to a} f(x)$ exist.

Solution:

Given $f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \le x \le 1 \\ 2-x, & x \ge 1 \end{cases}$

$y = 1+x, \; x < -1$

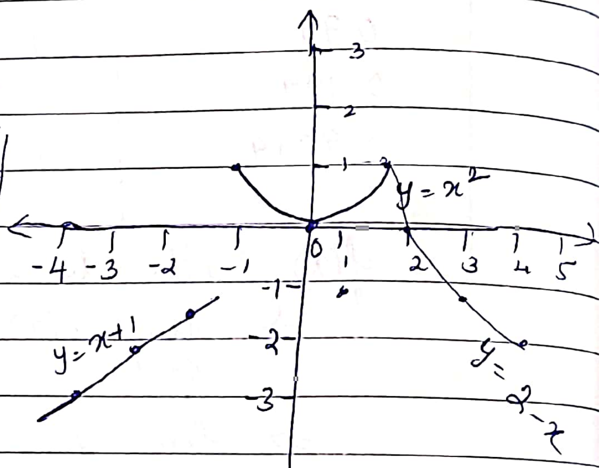
x	-2	-3	-4	-5
y	-1	-2	-3	-4

$y = x^2, \; -1 \le x \le 1$

x	-1	0	1
y	1	0	1

$2-x, \; x \ge 1$

x	1	2	3	4
y	1	0	-1	-2



From the graph $\lim_{x \to a} f(x)$ exist for all 'a' except at $a = -1$

$\Rightarrow$ right & left hand limit are different at $a = -1$

3 Find the $\lim_{x \to \infty} \frac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$

Solution: $\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$

$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x-2} + \sqrt{x+2})}{(\sqrt{3x-2} - \sqrt{x+2})(\sqrt{3x-2} + \sqrt{x+2})}$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)\left(\sqrt{3x-2} + \sqrt{x+2}\right)}{(3x-2)-(x+2)}$$

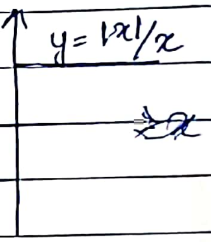
$$= \lim_{x \to 2} \frac{(x+2)(x-2)\left[\sqrt{3x-2} + \sqrt{x+2}\right)}{2x-4}$$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)\left(\sqrt{3x-2} + \sqrt{x+2}\right)}{2(x-2)}$$

$$= \frac{(2+2)\left(\sqrt{6-2} + \sqrt{4}\right)}{2}$$

$$= \frac{4[2+2]}{2} = 8$$

4. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ doesnot exist.



Solution:

Let $f(x) = \frac{|x|}{x}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} (x/x) = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^+} \left(-x/x\right) = \lim_{x \to 0^-} -1 = -1$$

$$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$$\Rightarrow \lim_{x \to 0} \frac{|x|}{x} \text{ does not exist.}$$

5. Find the domain where the function of f is continuous. Also, find the numbers at which the function f is discontinous where $f(x) = \begin{cases} 1+x^2, & x < 0 \\ 2-x; & 0 < x \leq 2 \\ (x-2)^2, & x > 2 \end{cases}$

Solution :

To find $f(0^-) = f(0) \neq f(0^+)$

At $x = 0$

$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} (1+x^2) = 1+1 = 1 \to (1)$

$f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1+x^2) = 1+0 = 1 \to (2)$

$f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2-x) = 2-0 = 2 \to (3)$

From (1), (2) & 3

we get $f(0^-) = f(0) \neq f(0^+)$

$\Rightarrow f$ is continuous at $x = 0$

At $x = 2$:

$f(2) = \lim_{x \to 2} f(x) = \lim_{x \to 2} (2-x)^2 = 0 \to (1)$

$f(2^-) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (2-x) = 2-2 = 0 \to (2)$

$f(2^+) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-2)^2 = 0 \to (3)$

From (1), (2) & (3)

$f(2^-) = f(2) = f(2^+)$

$\Rightarrow f$ is continuous at $x = 2$

$\therefore$ The domain of f is $(-\infty, 0) \cup (0, \infty)$

6. Use the squeeze theorem, find the value of

i) $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

Solution:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

{ The Squeeze theorem:

If $f(x) \le g(x) \le h(x)$ when x is near a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$ }.

we know that, $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$$\Rightarrow -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

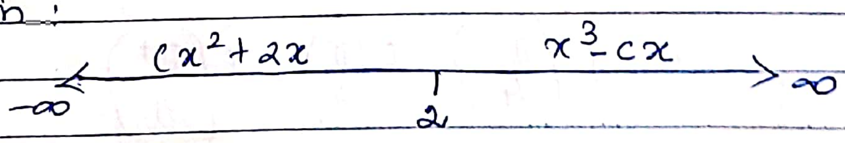
$$\lim_{x \to 0} (-x^2) = \lim_{x \to 0} (x^2) = 0$$

$$\Rightarrow \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$ by Squeeze theorem.

7. For what value of the constant c is the function f continuous at $(-\infty, \infty)$

$$f(x) = \begin{cases} cx^2 + 2x, & x < 2 \\ x^3 - cx, & x \ge 2 \end{cases}$$

Solution:



At $x = 2$,

Given: f is continuous $\rightarrow f(2^-) = f(2) = f(2^+)$ ↳ (A)

$f(2) = \lim_{x \to 2} f(x) = \lim_{x \to 2} (x^3 - cx) = 8 - 2c$ ↳ (1)

$f(2^-) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (cx^2 + 2x) = 4c + 4$ ↳ (2)

(A) ⇒ $8 - 2c = 4c + 4$ by (1) & (2)

$6c = 4$

$c = 2/3$

8. Show that f is continuous on $(-\infty, \infty)$

$$f(x) = \begin{cases} \sin x, & x < \frac{\pi}{4} \\ \cos x, & x \geq \frac{\pi}{4} \end{cases}$$

Solution:

At $x = \pi/4$ To prove: $f\left(\frac{\pi^-}{4}\right) = f\left(\frac{\pi}{4}\right) = f\left(\frac{\pi^+}{4}\right)$ ↳ (A)

$f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \cos x = \cos 45^\circ = \frac{1}{\sqrt{2}}$ → (1)

$f\left(\frac{\pi^-}{4}\right) = \lim_{x \to \frac{\pi^-}{4}} f(x) = \lim_{x \to \frac{\pi^-}{4}} \sin x = \sin 45^\circ = \frac{1}{\sqrt{2}}$ → (2)

$f\left(\frac{\pi^+}{4}\right) = \lim_{x \to \frac{\pi^+}{4}} f(x) = \lim_{x \to \frac{\pi^+}{4}} \cos x = \cos 45^\circ = \frac{1}{\sqrt{2}}$ → (3)

From (1), (2) & (3), we get (A)

i.e. $f\left(\frac{\pi^-}{4}\right) = f\left(\frac{\pi}{4}\right) = f\left(\frac{\pi^+}{4}\right)$

∴ f is continuous on $(-\infty, \infty)$

Q. i) Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where?

Solution:

Tangents are horizontal $\Rightarrow \frac{dy}{dx} = 0$

Given: $y = x^4 - 2x^2 + 2$

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2-1)$$

$$= 4x(x-1)(x+1)$$

$$dy/dx = 0 \Rightarrow 4x(x-1)(x+1) = 0$$

$$\Rightarrow x = 0, \; x = 1, \; x = -1$$

The curve will have horizontal tangents at $(0,2)$ $(1,1)$ $(-1,1)$

ii) Find y' for $\cos(xy) = 1 + \sin y$

Solution:

$$\cos(xy) = 1 + \sin y$$

$$-\sin(x,y)\left[x\frac{dy}{dx} + y(1)\right] = 0 + \cos y \frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - y\sin(xy) = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx}\left[\cos y + x\sin(xy)\right] = -y\sin(xy)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y\sin(xy)}{\cos y + \sin(xy)}$$

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10. Find the intervals of concavity and the inflection point

i) $f(x) = \sin x + \cos x$, $0 < x < 2\pi$

ii) $f(x) = x^4 - 2x^2 + 3$

Solution :

$f(x) = \sin x + \cos x$, $0 < x \leq 2\pi$

$f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x$

$\Rightarrow x = \pi/4, 5\pi/4$

a) critical points $x = \pi/4, 5\pi/4$

b)

Interval	Sign of f'	Behaviour of f
$0 < x < \pi/4$	+	increasing
$\pi/4 < x < 5\pi/4$	−	decreasing
$5\pi/4 < x < 2\pi$	+	increasing

c) The first derivatives test tell us that there is a local

i) Maximum at $\pi/4$, $f(\pi/4) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$

$= \sqrt{2}$

ii) Minimum at $5\pi/4$, $f(5\pi/4) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$

$= -\sqrt{2}$

d) $f''(x) = -\sin x - \cos x = -(\sin x + \cos x)$

$f''(x) = 0 \Rightarrow -(\sin x + \cos x) = 0$

$\Rightarrow \sin x = -\cos x$

$x \Rightarrow \frac{3\pi}{4}, \frac{7}{4}$

Intervals	Sign of f'	Behaviour of f
$0 < x < \frac{3\pi}{4}$	$-$	concave down
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$+$	Concave up
$\frac{7\pi}{4} < x < 2\pi$	$-$	Concave down

e) Inflection point are $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

Since $f\left(\frac{3\pi}{4}\right) = 0$, $f\left(\frac{7\pi}{4}\right) = 0$.

ii) Given

$f(x) = x^4 - 2x^2 + 3$

$f'(x) = 4x^3 - 4 = 4x(x^2-1) = 4x(x-1)(x+1)$

$f'(x) = 0 \Rightarrow 4x(x-1)(x+1) = 0$

$\Rightarrow x = 0;\ x = 1;\ x = -1$

a) Critical point are $x = 0,\ x = 1,\ x = -1$

b)

Interval	Sign of f'	Behavior of f
$-\infty < x < -1$	$-$	decreasing
$-1 < x < 0$	$+$	Increasing
$0 < x < 1$	$-$	decreasing
$1 < x < \infty$	$+$	increasing

c) The first derivative test tell us that

i) there is a local minimum of x at $x = \pm 1$

$f(\pm 1) = 2$

ii) there is a local maximum of x at $x = 0$

$f(0) = 3$

d) $f''(x) = 12x^2 - 4$

$f''(x) = 0 \Rightarrow 12x^2 - 4 = 0 \Rightarrow 3x^2 - 1 = 0$

$\Rightarrow 3x^2 = 1$

$\Rightarrow x^2 = 1/3 \Rightarrow x = \pm 1/\sqrt{3}$

Interval	Sign of f''	Behaviour of f
$-\infty < x < -1/\sqrt{3}$	+	Concave up
$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	−	Concave down
$\frac{1}{\sqrt{3}} < x < \infty$	+	Concave up

e) Inflection point are $\left(\pm \frac{1}{\sqrt{3}}, \frac{22}{9}\right)$

Since $f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{22}{9}$.

UNIT III – FUNCTIONS OF SEVERAL VARIA BLES	
PART – A	
1	**If $u = xy - 2yz + z^2$, then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.**
	Solution: $u = xy - 2yz + z^2$ $\frac{\partial u}{\partial x} = y; \quad \frac{\partial u}{\partial y} = x - 2z; \quad \frac{\partial u}{\partial z} = -2y + 2z;$ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = y + x - 2z - 2y + 2z = x - y$
3	**Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when $u(x,y) = x^y + y^x$** **(NOV/DEC 2020)**
	Solution: Given $u(x,y) = x^y + y^x$ $\frac{\partial u}{\partial x} = yx^{y-1} + y^x \log(y)$ $\quad \left(\because \frac{d}{dx}(a^x) = a^x \log a, \text{ but } \frac{d}{dx}(x^a) = ax^{a-1}\right)$ $\frac{\partial u}{\partial y} = x^y \log(x) + xy^{x-1}$
4	**Verify the Euler's theorem for the function $u = x^2 + y^2 + 2xy$.** **(NOV/DEC 2019)**
	Solution: **Euler's theorem:** If $u(x,y)$ is homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ $u(x,y) = x^2 + y^2 + 2xy$ $u(x,y) = x^2 + y^2 + 2xy \Rightarrow u(tx,ty) = (tx)^2 + (ty)^2 + 2(tx)(ty)$ $u(tx,ty) = t^2x^2 + t^2y^2 + 2t^2xy = t^2\left(x^2 + y^2 + 2xy\right)$ i.e t^2u $\therefore$ The degree of $u(x,y) = x^2 + y^2 + 2xy$ is 2. $\quad \therefore$ Degree = n = 2. $\because$ u is homogeneous function of 2nd degree in x and y, by Euler's theorem we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ $(i.e)\, x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\left(x^2 + y^2 + 2xy\right)$ Now, $u = x^2 + y^2 + 2xy$ $x\frac{\partial u}{\partial x} = x(2x + 2y) = 2x^2 + 2xy$, $\quad y\frac{\partial u}{\partial y} = y(2y + 2x) = 2y^2 + 2xy$ $L.H.S = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2 + 2xy + 2y^2 + 2xy = 2x^2 + 2y^2 + 4xy = 2(x^2 + y^2 + 2xy) = R.H.S$
5	**If $z = xf\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ using Euler's theorem** **(NOV / DEC 2020)**
	Solution: Given $z = xf\left(\frac{y}{x}\right)$.

i.e., $z(x,y) = xf\left(\frac{y}{x}\right)$

$$z(tx,ty) = txf\left(\frac{ty}{tx}\right) = txf\left(\frac{y}{x}\right) = t.z(x,y)$$

$$z(tx,ty) = t^1 z(x,y)$$

$\therefore z(x,y)$ is a homogeneous function of degree 1 in x and y. $\therefore$ Degree = n = 1.

Euler's theorem: If $u(x,y)$ is homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

$\because$ z is a homogeneous function of 1st degree in x and y, by Euler's theorem , we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1(z) = z$$

6 **If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.**

Solution: $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f\ (x,y)$

$$f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}\left(\sqrt{x}+\sqrt{y}\right)} = \frac{\sqrt{t}(x+y)}{\left(\sqrt{x}+\sqrt{y}\right)} = t^{1/2} f(x,y)$$

$f = \cos u$ is a homogeneous function of degree $n = \frac{1}{2}$ in x and y.

Therefore, by Euler's theorem we get,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial(\cos u)}{\partial x} + y\frac{\partial(\cos u)}{\partial y} = \frac{1}{2}\cos u$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$-\sin u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u$$

7 **If $u = x^3 + y^3$ and $x = at^2$, $y = 2at$, then find $\frac{du}{dt}$.** **(APR/MAY 2019)**

Solution: Given $u = x^3 + y^3$ and $x = at^2$, $y = 2at$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$u = x^3 + y^3$	$x = at^2$	$y = 2at$

$\frac{\partial u}{\partial x}=3x^2$	$\frac{\partial u}{\partial y}=3y^2$	$\frac{dx}{dt}=2at$	$\frac{dy}{dt}=2a$

$$\frac{du}{dt}=(3x^2)(2at)+(3y^2)(2a)$$

$$=(3(\mathrm{at}^2)^2)(2at)+(3)(2\,\mathrm{at})^2(2a) \qquad \left(\because x=at^2,\ y=2at\right)$$

$$=6a^3t^5+24a^3t^2=6a^3t^2(\mathrm{t}^3+4)$$

8 **Find $\frac{dy}{dx}$, if $x^3+y^3=6xy$.**

Solution:

$$x^3+y^3-6xy=0$$

$$Let\ \ f(x,y)=x^3+y^3-6xy$$

$$f_x=\frac{\partial f}{\partial x}=3x^2-6y;\quad f_y=\frac{\partial f}{\partial y}=3y^2-6x$$

$$\frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-(3x^2-6y)}{3y^2-6x}=\frac{6y-3x^2}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$

9 **What is the derivative of u with respect to x, for $u=x^2y^3$, where $2\sin x-3y=0$?**

Solution:

$$2\sin x-3y=0$$

$$2\cos x-3\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{2\cos x}{3}$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}$$

$$=2xy^3+3x^2y^2.\frac{2\cos x}{3}$$

$$=2xy^3+2x^2y^2\cos x$$

$$=2xy^2(y+x\cos x)$$

10 **State the properties of Jacobian.** **(NOV/DEC 2018)**

Solution:

(i) If *u* and *v* are functions of *r* and *s*, *r* and *s* are functions of *x* and *y* then, $\frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(x,y)}$

(ii) If *u* and *v* are functions of *x* and *y* then, $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}=1$ (i.e) $JJ'=1$

(iii) If u, v, w are functionally dependent functions of three independent variable x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$

11 **If $x=u^2-v^2$ and $y=2uv$, find the Jacobian of x and y with respect to u and v.**

(APR/MAY 2019)

Solution: $x=u^2-v^2\Rightarrow\frac{\partial x}{\partial u}=2u;\ \frac{\partial x}{\partial v}=-2v$ and $y=2uv\Rightarrow\frac{\partial y}{\partial u}=2v;\ \frac{\partial y}{\partial v}=2u$

	$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2u & -2v\\ 2v & 2u\end{vmatrix}=4(u^2+v^2)$
12	**If $x=uv$ and $y=\frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.** **(JAN 2018)**
	Solution: $x=uv\Rightarrow\frac{\partial x}{\partial u}=v;\ \frac{\partial x}{\partial v}=u$ and $y=\frac{u}{v}\Rightarrow\frac{\partial y}{\partial u}=\frac{1}{v};\ \frac{\partial y}{\partial v}=-\frac{u}{v^2}$ $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v & u\\ \frac{1}{v} & \frac{-u}{v^2}\end{vmatrix}=v\left(\frac{-u}{v^2}\right)-u\left(\frac{1}{v}\right)=\frac{-u}{v}-\frac{u}{v}=\frac{-2u}{v}$
13	**If $x=r\cos\theta,\ y=r\sin\theta$ *then find* $\frac{\partial(r,\theta)}{\partial(x,y)}$** **(JAN 2018) & (NOV/DEC 2019)**
	Solution: $x=r\cos\theta\Rightarrow\frac{\partial x}{\partial r}=\cos\theta;\ \frac{\partial x}{\partial\theta}=r(-\sin\theta)$ and $y=r\sin\theta\Rightarrow\frac{\partial y}{\partial r}=\sin\theta;\ \frac{\partial y}{\partial\theta}=r\cos\theta$ $\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$ $\therefore\frac{\partial(r,\theta)}{\partial(x,y)}=\frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}}=\frac{1}{r}$
14	**Find Taylor's series expansion of x^y near the point (1,1) up to first degree terms.**
	Solution: The Taylor's series expansion is given by $f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+\ldots$ $f(x,y)=x^y$; $f(1,1)=1$ $f_x(x,y)=yx^{y-1}$; $f_x(1,1)=1$ $f_y(x,y)=x^y\log x$; $f_y(1,1)=0$ $f(x,y)=1+(x-1)(1)+(y-1)(0)$ $=1+x-1$ $x^y=x$
15	**Obtain Taylor's series expansion of e^{x+y} in powers of x and y up to first degree terms.**
	Solution: The Taylor's series expansion is given by $f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+\ldots$ $f(x,y)=e^{x+y}$; $f(0,0)=1$ $f_x(x,y)=e^{x+y}$; $f_x(0,0)=1$ $f_y(x,y)=e^{x+y}$; $f_y(0,0)=1$ $f(x,y)=f(0,0)+\left[(x-0)f_x(0,0)+(y-0)f_y(0,0)\right]$

	$=1+x(1)+y(1)$ $e^{x+y}=1+x+y$
16	**State the conditions for maxima and minima of $f(x,y)$.**
	Solution: If $f_x(a,b)=0,\ f_y(a,b)=0$ and $f_{xx}(a,b)=A,\ f_{xy}(a,b)=B,\ f_{yy}(a,b)=C$ then (i) $f(x,y)$ has maximum value at (a,b) if $AC-B^2>0$ and A< 0 or C< 0 (ii) $f(x,y)$ has minimum value at (a,b) if $AC-B^2>0$ and A > 0 or C> 0
17	**Find the maxima and minima of $f(x,y)=x^2+y^2+6x+4y+12$.**
	Solution: Given $f(x,y)=x^2+y^2+6x+4y+12$ $f_x=2x+6=0\Rightarrow x=-3;\quad f_y=2y+4=0\Rightarrow y=-2$. The stationary point is (-3, -2). $A=f_{xx}=2;\ B=f_{xy}=0\ ;\ C=f_{yy}=2,$ $AC-B^2=4>0$ and $A>0$. $\therefore f$ is minimum at (−3, -2) and the minimum value is $f(-3,-2)=(-3)^2+(-2)^2+6(-3\ )+4(-2)+12=25-26=-1$.
18	**Find the possible extreme point of $f(x,y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}$.**
	Solution: $f(x,y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}$ $\frac{\partial f}{\partial x}=2x-\frac{2}{x^2};$ $\frac{\partial f}{\partial x}=0\Rightarrow 2x-\frac{2}{x^2}=0;\ 2x=\frac{2}{x^2};\ x^3=1\ \Rightarrow x=1$ $\frac{\partial f}{\partial y}=2y-\frac{2}{y^2};$ $\frac{\partial f}{\partial y}=0\ \Rightarrow 2y-\frac{2}{y^2}=0;\ 2y=\frac{2}{y^2};\ y^3=1\ \Rightarrow y=1$ $\therefore$ The possible extreme point is $(1,1)$.
19	**A rectangular box open at the top is to have a maximum capacity whose surface area is 648 square centimeters. Formulate the maximization function to find the dimensions of the box.**
	Solution: Let x,y,z be the length,breadth and height of the box. volume $=length\times breadth\times height=xyz$ $\qquad$ ($Volume\ to\ be$ max$imized$) $Let\ f(x,y,z)=xyz$ Since the rectangular box is of open at the top, Surface area on the top is zero. $\therefore$ Total surface area of the box $=xy+2yz+2zx=648$ $\quad$ ($given$) $Let\ g(x,y,z)=xy+2yz+2zx-648$ The optimization function to find the dimensions of the box is $F(x,y,z)=f(x,y,z)+\lambda g(x,y,z)$ where λ is langrange multiplier.

	$Hence,\ F(x,y,z) = xyz + \lambda\left(xy + 2yz + 2zx - 648\right)$
20	**Formulate the optimization function to find the volume of the largest rectangular solid which can be inscribed in an ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
	Solution: Let the sides of the rectangular box be $2x,\ 2y,\ 2z$. The largest rectangular parallelepiped inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ will have its corners on the ellipsoid and its sides parallel to the coordinate plane. Hence its corners are $(\pm x, \pm y, \pm z)$. $\text{Volume} \quad V = 2x \times 2y \times 2z$ $= 8xyz$ $\text{Let} \quad f = 8xyz$ $g = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$ The optimization function to find the dimensions of the box is $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z) \quad (i.e.) \quad F = f + \lambda g$ $F(x,y,z) = (8xyz) + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$ where λ is langrange multiplier.

1. **If $x = uv, y = \frac{u}{v}$ prove $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$**

Solution:

Let $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \dots (1)$

Given $x = uv, y = \frac{u}{v}$

$\frac{\partial x}{\partial u} = v, \frac{\partial y}{\partial u} = \frac{1}{v}$

$\frac{\partial x}{\partial v} = u \quad , \quad \frac{\partial x}{\partial u} = \frac{-u}{v^2}$

$(1) \Rightarrow \quad J = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$

$= \frac{-uv}{v^2} - \frac{u}{v} = \frac{-2u}{v}$

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \dots (2)$

$x = uv, \; y = \frac{u}{v} \;\Rightarrow\; u = vy$

$x = vyv \;\Rightarrow x = v^2 y \;\Rightarrow v^2 = \frac{x}{y} \;\Rightarrow v = \frac{\sqrt{x}}{\sqrt{y}}$

$x = uv \;\Rightarrow\; u = \frac{x}{v} \Rightarrow\; u = \frac{x\sqrt{y}}{\sqrt{x}} = \sqrt{x}\sqrt{y}$

$Now \Rightarrow \; u = \sqrt{x}\sqrt{y}, \; v = \frac{\sqrt{x}}{\sqrt{y}}$

$\frac{\partial u}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}}, \frac{\partial v}{\partial x} = \frac{1}{\sqrt{y}} \frac{1}{2\sqrt{x}}$

$\frac{\partial u}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}}, \frac{\partial v}{\partial y} = \sqrt{x}\left(\frac{-1}{y}\right)\frac{1}{2\sqrt{y}}$

$(2) \Rightarrow J' = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{y}\sqrt{x}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$

$= -\frac{1}{4y} - \frac{1}{4y} = -\frac{2}{4y} = -\frac{1}{2y}$

$= -\frac{v}{2u} \qquad (\because y = \frac{u}{v})$

here $J = \frac{-2u}{v}$ and $J' = -\frac{v}{2u}$

$\therefore \; JJ' = \frac{-2u}{v} \times (-\frac{v}{2u}) = 1$

Hence proved.

2. **If $x + y + z = u, y + z = uv, z = uvw$ prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$**

Solution:

Given $x + y + z = u,\ y + z = uv,\ z = uvw$

$x + uv = u,\ y + uvw = uv,\ z = uvw$

$x = u - uv,\ y = uv - uvw,\ z = uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial z} \end{vmatrix} \ldots(1)$$

$x = u - uv,\ y = uv - uvw,\ z = uvw$

$\frac{\partial x}{\partial u} = 1 - v,\ \frac{\partial y}{\partial u} = v - vw,\ \frac{\partial z}{\partial u} = vw$

$\frac{\partial x}{\partial v} = -u,\ \frac{\partial y}{\partial v} = u - uw,\ \frac{\partial z}{\partial v} = uw$

$\frac{\partial x}{\partial w} = 0,\ \frac{\partial y}{\partial w} = -uv,\ \frac{\partial z}{\partial w} = uv$

$$(1) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 - v & -u & 0 \\ v - vw & u - uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1 - v)(u^2v - u^2vw + u^2vw) + u(uv^2 - uv^2w + uv^2w) +$$

$$0(uvw - uvw^2 - uvw + uvw^2$$

$$= (1 - v)u^2v + u^2v^2$$

$$= u^2v - u^2v^2 + u^2v^2$$

$$= u^2v$$

Hence proved.

3. **If $p = 3x + 2y - z$, $q = x - 2y = z$, $r = x + 2y - z$ prove that p , q , r are functionally dependent.**

Solution:

To prove p , q , r are functionally dependent.

i.e) To prove $\frac{\partial(p,q,r)}{\partial(x,y,z)} = 0$

$$\text{Consider } \frac{\partial(p,q,r)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

= 3(2-2) – 2(-1 -1) -1(2+2)

= 4 – 4

= 0

$\therefore p$, q , r are functionally dependent.

4. **Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$**

Solution:

Given $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3 \quad ; \; f_y = 3y^2 - 12$

$f_{xx} = 6x = A \, , \; f_{xy} = 0 = B \; , \; f_{yy} = 6y = C$

To find the stationary points.

$f_x = 0$	$f_x = 0$
$3x^2 - 3 = 0$ $x^2 - 1 = 0$ $x = \pm 1$	$3y^2 - 12 = 0$ $y^2 - 4 = 0$ $y = \pm 2$

$\therefore$ Stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6 > 0$	$6 > 0$	$-6 < 0$	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. point	Saddle point	Saddle point	Max. point

$\therefore$ Maximum value of $f(x,y)$ is

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$

$$= -1 - 8 + 3 + 24 + 20 = 38$$

Minimum value of $f(x,y)$ is

$$f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2$$

5. **Find the maxima and minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$**

Solution:

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$f_x = 4x^3 - 4x + 4y \quad ; \quad f_y = 4y^3 + 4x - 4y$$

$$f_{xx} = 12x^2 - 4 = A, \; f_{xy} = 4 = B, \; f_{yy} = 12y^2 - 4 = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$

$4x^3 - 4x + 4y = 0$ $x^3 - x + y = 0 \ldots \; (1)$	$4y^3 + 4x - 4y = 0$ $y^3 + x - y = 0 \ldots \; (2)$

$$(1) + (2) \Rightarrow x^3 + y^3 = 0 \quad \Rightarrow x^3 = -y^3 \quad \Rightarrow y = -x$$

$$(1) \Rightarrow x^3 - x - x = 0 \quad \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$$

$$\Rightarrow x = 0 \; (or) \; (x^2 - 2) = 0$$

$$\Rightarrow x = 0 \; (or) \; x = \pm\sqrt{2}$$

$\therefore$ The stationary points are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

	$(0,0)$	$(\sqrt{2}, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$
$A = 12x^2 - 4$	$-4 < 0$	$20 > 0$	$20 > 0$
$B = 4$	4	4	4
$C = 12y^2 - 4$	-4	20	20
$AC - B^2$	0	$384 > 0$	$384 > 0$
Conclusion	Cannot be an extreme point	Minimum point	Minimum point

Minimum at $(\sqrt{2}, -\sqrt{2})$

$$= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 4 - 8 - 4$$

$$= -8$$

Minimum at $(-\sqrt{2}, \sqrt{2})$

$$= (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})\sqrt{2} - 2(\sqrt{2})^2$$

$$= 4 + 4 - 8 - 4 - 4 = -8$$

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6. **Obtain terms up to the third degree in the Taylor series expansion of $e^x siny$ about the point $\left(1, \frac{\pi}{2}\right)$**

Solution:

Function	**Value at $(1, \frac{\pi}{2})$**
$f(x, y) = e^x siny$	$f = e$
$f_x = e^x siny$ $f_y = e^x cosy$	$f_x = e$ $f_y = 0$
$f_{xx} = e^x siny$ $f_{xy} = e^x \cos y$ $f_{yy} = -e^x \text{siny}$	$f_{xx} = e$ $f_{xy} = 0$ $f_{yy} = -e$
$f_{xxx} = e^x siny$ $f_{xxy} = e^x \cos y$ $f_{xyy} = -e^x siny$ $f_{yyy} = -e^x \text{cosy}$	$f_{xxx} = e$ $f_{xxy} = 0$ $f_{xyy} = -e$ $f_{yyy} = 0$

By Taylor's theorem

$$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$

$$+\frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$

$$+\frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] + \cdots$$

Put $\boldsymbol{a = 1, b = \frac{\pi}{2}}$

$$f(x,y) = e + \frac{1}{1!}\left[(x-1)e + \left(y-\frac{\pi}{2}\right)(0)\right] +$$

$$\frac{1}{2!}\left[(x-1)^2 e + 2(x-1)\left(y-\frac{\pi}{2}\right)(0) + \left(y-\frac{\pi}{2}\right)^2(-e)\right] +$$

$$\frac{1}{3!}\left[(x-1)^3 e + 3(x-1)^2\left(y-\frac{\pi}{2}\right)(0) + 3(x-1)\left(y-\frac{\pi}{2}\right)^2(-e) + \left(y-\frac{\pi}{2}\right)^3(0)\right] + \cdots$$

$$f(x,y) = e + \frac{1}{1!}(x-1)e + \frac{1}{2!}\left[(x-1)^2 e + \left(y-\frac{\pi}{2}\right)^2(-e)\right]$$

$$+\frac{1}{3!}\left[(x-1)^3 e - 3e(x-1)\left(y-\frac{\pi}{2}\right)^2\right] + \cdots$$

7. **Expand $\boldsymbol{e^x \log(1+y)}$ in powers of x and y upto terms of third degree.**

Solution:

Function	**Value at (0 ,0)**
$f(x,y) = e^x \log(1+y)$	$f = 0$
$f_x = e^x \log(1+y)$ $f_y = e^x \frac{1}{1+y}$	$f_x = 0$ $f_y = 1$
$f_{xx} = e^x \log(1+y)$ $f_{xy} = e^x \frac{1}{1+y}$ $f_{yy} = -e^x \frac{1}{(1+y)^2}$	$f_{xx} = 0$ $f_{xy} = 1$ $f_{yy} = -1$

$f_{xxx} = e^x \log(1+y)$ $f_{xxy} = e^x \frac{1}{1+y}$ $f_{xyy} = -e^x \frac{1}{(1+y)^2}$ $f_{yyy} = 2\, e^x \frac{1}{(1+y)^3}$	$f_{xxx} = 0$ $f_{xxy} = 1$ $f_{xyy} = -1$ $f_{yyy} = 2$

By Taylor's theorem

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

Put $\boldsymbol{a} = \mathbf{0}, \boldsymbol{b} = \mathbf{0}$

$$f(x,y) = 0 + \frac{1}{1!}[(x)(0) + (y)(1)] + \frac{1}{2!}[(x)^2(0) + 2(x)(y)(1) + (y)^2(-1)]$$

$$+\frac{1}{3!}[(x)^3(0) + 3(x)^2(y)(1) + 3(x)(y)^2(-1) + (y)^3(2)] + \cdots$$

$$= y + \frac{2xy - y^2}{2!} + \frac{3x^2y - 3xy^2 + 2y^3}{3!} + \cdots$$

8. Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ up to the third degree terms.

[A.U A/M 2014) (A.U Jan. 2012]

[**Ans:** $6 + [(x+2)(-9) + (y-1)(4)] + \frac{1}{2!}[(x+2)^2(6) - 20(x+2)(y-1) + (y-1)^2(-4)] + \frac{1}{3!}[(x+2)^2(y-1)(24) + (x+2)(y-1)^2(6)] + \cdots$

9. **If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$ and $v = 2xy$ then prove that**

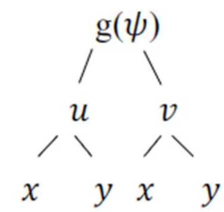
$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right)$$

Solution:

Given $u = x^2 - y^2 , v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$



$$\frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v}\frac{\partial v}{\partial x} \text{ ... (1)}$$

$$\frac{\partial g}{\partial y} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \psi}{\partial v} \text{ ... (2)}$$

$$(1) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u}2x + \frac{\partial \psi}{\partial v}2y$$

$$\frac{\partial g}{\partial x} = 2\left(x\frac{\partial \psi}{\partial u} + y\frac{\partial \psi}{\partial v}\right)$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right) \ 2\left(x\frac{\partial \psi}{\partial u} + y\frac{\partial \psi}{\partial v}\right)$$

$$= 4\left[x^2\frac{\partial^2 \psi}{\partial u^2} + 2xy\frac{\partial^2 \psi}{\partial u \partial v} + y^2\frac{\partial^2 \psi}{\partial v^2}\right] \text{ ... (3)}$$

$$(2) \Rightarrow \frac{\partial g}{\partial y} = \frac{\partial \psi}{\partial u}(-2y) + \frac{\partial \psi}{\partial v}2x$$

$$\frac{\partial g}{\partial y} = 2\left(-y\frac{\partial \psi}{\partial u} + x\frac{\partial \psi}{\partial v}\right)$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right) \ 2\left(-y\frac{\partial \psi}{\partial u} + x\frac{\partial \psi}{\partial v}\right)$$

$$= 4\left[y^2\frac{\partial^2 \psi}{\partial u^2} - 2xy\frac{\partial^2 \psi}{\partial u \partial v} + x^2\frac{\partial^2 \psi}{\partial v^2}\right] \text{ ... (4)}$$

$$(3) + (4) \Rightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4\left\{(x^2 + y^2)\frac{\partial^2 \psi}{\partial u^2} + (x^2 + y^2)\frac{\partial^2 \psi}{\partial u^2}\right\}$$

$$= 4(x^2 + y^2)\left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right)$$

Hence proved.

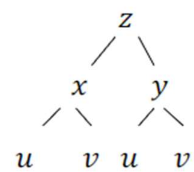
10. **If z is a function of x and y where $x = e^{u} + e^{-v}$ and $y = e^{-u} - e^{v}$, show that** $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution:

Given $x = e^{u} + e^{-v}$; $y = e^{-u} - e^{v}$

$$\frac{\partial x}{\partial u} = e^{u}, \qquad \frac{\partial y}{\partial u} = -e^{-u}$$

$$\frac{\partial x}{\partial v} = -e^{-v}, \qquad \frac{\partial y}{\partial v} = -e^{v}$$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \dots(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \quad \dots(2)$$

$$(1) \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}e^{u} - \frac{\partial z}{\partial y}e^{-u}$$

$$(2) \Rightarrow \frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}e^{v}$$

$$(1) - (2) \Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Hence proved.

11. **A rectangular box open at the top, is to have a volume of 32cc. find the dimensions of the box that requires the least material for its construction.**

Solution:

Let x, y, z be the length, breadth and height of the box.

Surface area = $xy + 2yz + 2zx$

Volume = $xyz = 32$

Let the auxiliary function F be

$F(x, y, z, \lambda) = (xy + 2yz + 2zx) + \lambda(xyz - 32)$

Where λ is Lagrange Multiplier

$\frac{\partial F}{\partial x} = y + 2z + \lambda yz$	$\frac{\partial F}{\partial y} = x + 2z + \lambda xz$	$\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$

When F is extremum

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$y + 2z + \lambda yz = 0$ $\Rightarrow y + 2z = -\lambda yz$	$x + 2z + \lambda zx = 0$ $\Rightarrow x + 2z = -\lambda yz$	$2x + 2y + \lambda xy = 0$ $\Rightarrow 2x + 2y = -\lambda xy$

$\Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda$ ...(1)	$\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda$ ...(2)	$\Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda$ ...(3)

From (1) and (2), we get $\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$ $\frac{2}{y} = \frac{2}{x}$ $x = y$ ...(4)	From (2) and (3), we get $\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$ $\frac{1}{z} = \frac{2}{y}$ $y = 2z$ ...(5)

From (4) and (5), we get

$$x = y = 2z$$

Volume = $xyz = 32 \Rightarrow (2z)(2z)z = 32 \Rightarrow 4z^3 = 32$

$$z^3 = \frac{32}{4} = 8 \Rightarrow z = 2 \quad i.e.\ x = 4,\ y = 4, z = 2$$

$\therefore$ Cost minimum when $x = 4, y = 4, z = 2$

12. **Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre**

Solution:

Let x, y, z be the length, breadth and height of the box.

Surface area = $xy + 2yz + 2zx = 432$

Volume $= xyz$

Let the auxiliary function F be

$$F(, y, z, \lambda) = xyz + \lambda\,(xy + 2yz + 2zx - 432)$$

$\frac{\partial F}{\partial x} = yz + \lambda\,(y + 2z)$	$\frac{\partial F}{\partial y} = xz + \lambda\,(x + 2z)$	$\frac{\partial F}{\partial z} = xy + \lambda\,(2y + 2x)$

To find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$yz + \lambda\,(y + 2z) = 0$ $\Rightarrow\ yz = -\lambda\,(y + 2z)$ $\Rightarrow\ \frac{y + 2z}{yz} = \frac{-1}{\lambda}$ $\Rightarrow\ \frac{1}{z} + \frac{2}{y} = -\frac{1}{\lambda}$ ...(1)	$xz + \lambda\,(x + 2z) = 0$ $\Rightarrow\ xz = -\lambda\,(x + 2z)$ $\Rightarrow\ \frac{x + 2z}{xz} = \frac{-1}{\lambda}$ $\Rightarrow\ \frac{1}{z} + \frac{2}{x} = -\frac{1}{\lambda}$ ...(2)	$xy + \lambda\,(2y + 2x) = 0$ $\Rightarrow\ xy = -\lambda\,(2y + 2x)$ $\Rightarrow\ \frac{2y + 2x}{xy} = \frac{-1}{\lambda}$ $\Rightarrow\ \frac{2}{x} + \frac{2}{y} = -\frac{1}{\lambda}$ ...(3)

From (1) & (2), *we get* $\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$ $\Rightarrow \frac{2}{y} = \frac{2}{x}$ $\Rightarrow x = y$ ...(4)	*From* (2) & (3), *we get* $\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$ $\Rightarrow \frac{1}{z} = \frac{2}{y}$ $\Rightarrow y = 2z$ ...(5)

From (4) & (5), we get $x = y = 2z$

Surface area = $xy + 2yz + 2zx = 432$

$$(2z)(2z) + 2(2z)z + 2z(2z) = 432$$

$$4z^2 + 4z^2 + 4z^2 = 432$$

$$12z^2 = 432$$

$$z^2 = 36 \therefore z = 6$$

$$\therefore x = 12, y = 12, \quad z = 6 \qquad by\ (6)$$

Thus, the dimension of the box is 12, 12, 6.

Maximum volume = 12 x 12 x 6 = 864 cubic metres.

13. **The temperature $u(x, y, z)$ at any point in space is $u = 400xyz^2$. Find the highest temperature on surface of the sphere $x^2 + y^2 + z^2 = 1$.**

Solution:

$$u = f = 400xyz^2 \quad \dots (A)$$

$$\emptyset = x^2 + y^2 + z^2 - 1 = 0 \quad \dots (B)$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

To

$\frac{\partial F}{\partial x} = 400yz^2 + \lambda(2x)$	$\frac{\partial F}{\partial y} = 400xz^2 + \lambda(2y)$	$\frac{\partial F}{\partial z} = 800xyz + \lambda(2z)$

find the stationary value.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$400yz^2 + \lambda(2x) = 0$ $400yz^2 = -\lambda(2x)$ $\frac{200yz^2}{x} = -\lambda \quad \dots (1)$	$400xz^2 + \lambda(2y) = 0$ $400xz^2 = -\lambda(2y)$ $\frac{200xz^2}{y} = -\lambda \quad \dots (2)$	$800xyz + \lambda(2z) = 0$ $800xyz = -\lambda(2z)$ $400xy = -\lambda \quad \dots (3)$

From (1) & (2), we get $y^2 = x^2 \dots (4)$

From (2) & (3), we get $z^2 = 2y^2 \dots (5)$

From (4) & (5), we get

$$x^2 = y^2 = \frac{1}{2}z^2 \quad \dots (6)$$

$$(B) \Rightarrow \frac{1}{2}z^2 + \frac{1}{2}z^2 + z^2 - 1 = 0$$

$$\Rightarrow 2z^2 = 1$$

$$\Rightarrow z^2 = \frac{1}{2} \Rightarrow z = \pm\frac{1}{2}$$

(6) $\Rightarrow x^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \quad \Rightarrow x = \pm\frac{1}{2}$

(6) $\Rightarrow y^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \quad \Rightarrow y = \pm\frac{1}{2}$

$\therefore\ u = 400xyz^2$, we select x, y, z to be positive

$$\Rightarrow u = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow u = 50$$

$\therefore$ Maximum temperature is 50

UNIT IV

INTEGRAL CALCULUS

Problems:

1. Evaluate $\int_0^3 (x^2 - 2x)dx$ by using Riemann sum by taking right end points as the sample points.

Solution:

Take n subintervals, we have $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{6}{n}, x_3 = \frac{9}{n}, \ldots, x_i = \frac{3i}{n}, \ldots\ x_n = \frac{3x}{n}$

Since we are using right end points.

$$\int_0^3 (x^2 - 2x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[(\frac{3i}{n})^2 - 2(\frac{3i}{n})\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[\frac{9}{n^2}i^2 - \frac{6}{n}i\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2}\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{6n^3} n^3 \left[1 + \frac{1}{n}\right]\left[2 + \frac{1}{n}\right] - \lim_{n\to\infty} \frac{9}{n^2}\ n^2 \frac{n(n+1)}{2}\left[1 + \frac{1}{n}\right]$$

$$= \frac{27}{16}(1)(2) - 9(1)$$

$$= 9 - 9 = 0$$

2. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{sinx + cosx}dx$

Solution

Let I = $\int_0^{\frac{\pi}{2}} \frac{\cos x}{sinx + cosx}dx$ ….(1)

$$= \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2} - x)}{sin(\frac{\pi}{2} - x) + cos(\frac{\pi}{2} - x)}dx$$

$$\therefore \text{I} = \int_0^{\frac{\pi}{2}} \frac{\sin x}{sinx+cosx} dx \quad \ldots.(2)$$

$$(1) + (2) \Longrightarrow 2\text{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x}{sinx+cosx} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{sinx+cosx} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{sinx+cosx} + \frac{\sin x}{sinx+cosx} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x+cosx}{sinx+cosx} dx$$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 0$$

$$\therefore \text{I} = \frac{\pi}{2} \; \frac{1}{2}$$

$$= \frac{\pi}{4}$$

3. Evaluate $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

Solution:

$$\text{Let I} = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right) dx \ \ldots\ldots(1)$$

$$= \int_0^1 \log\left(\frac{1-(1-x)}{(1-x)}\right) dx$$

$$\therefore \text{I} = \int_0^1 \log\left(\frac{x}{1-x}\right) dx \quad \ldots..(2)$$

$$(1) + (2) \Longrightarrow 2\text{I} = \int_0^1 \log\left(\frac{1-x}{x}\right) dx + \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \left[\log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right)\right] dx$$

$$= \int_0^1 \left[\log\left(\frac{1-x}{x}\right)\left(\frac{x}{1-x}\right)\right] dx$$

$$= \int_0^1 \log 1 \ dx$$

$= \int_0^1 \; dx = 0$

$\therefore 2\,I = 0$

$I = 0$

4. Using integration by parts, evaluate $\int \frac{(logx)^2}{x^2}\; dx$

Solution:

Let $I = \int \frac{(logx)^2}{x^2}\; dx$

$\int u\, dv = uv - \; vdu$

$u = (logx)^2$

$du = 2(logx)\left(\frac{1}{x}\right) dx$

$dv = \frac{1}{x^2}\; dx$

$\int dv = \int \frac{1}{x^2} dx$

$v = -\frac{1}{x}$

$\therefore I = (logx)^2 \left(\frac{-1}{x}\right) - \int \left(\frac{-1}{x}\right) 2\;(logx)\left(\frac{1}{x}\right) dx$

$= -\frac{1}{x}(logx)^2 + 2logx\left(\frac{1}{x^2}\right) dx$

Again let $u = logx$

$du = \frac{1}{x} dx$

$dv = \frac{1}{x^2}\; dx$

$v = -\frac{1}{x}$

$I = \left(\frac{-1}{x}\right)(logx)^2 + 2\left[(logx)\left(\frac{1}{x}\right) - \int \left(\frac{-1}{x}\right)\frac{1}{x}\; dx\right]$

$= \left(\frac{-1}{x}\right)(logx)^2 - 2\left(\frac{1}{x}\right) logx + 2\int \frac{1}{x^2} dx$

$= \left(\frac{-1}{x}\right)(logx)^2 - 2\left(\frac{1}{x}\right) logx - 2\left(\frac{1}{x}\right) + c$

5. Evaluate $\int tan^{-1}xdx$. Also find $\int_0^1 tan^{-1}xdx$

Solution:

Let $u = tan^{-1}x$

$du = \frac{1}{1+x^2}dx$

$dv = dx$

$v = \int dx = x$

$\int u\,dv = uv - \ vdu$

$\int tan^{-1}xdx = xtan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx$

$= xtan^{-1}x - \int \left(\frac{1}{1+x^2}\right)dx$ .....(1)

Take $\int \left(\frac{1}{1+x^2}\right)dx$

Put $t = \ 1 + x^2$

$dt = 2xdx$

$\int \left(\frac{x}{1+x^2}\right)dx = \int \frac{1}{t}\frac{1}{2}dt$

$= \frac{1}{2}\int \frac{1}{t}dt$

$= \frac{1}{2}logt$

$= \frac{1}{2}log(1 + x^2)$

(1)$\Rightarrow \int tan^{-1}xdx = xtan^{-1}x - \frac{1}{2}log(1 + x^2) + c$......(2)

To find $\int_0^1 tan^{-1}xdx$

(2) $\Rightarrow \int_0^1 tan^{-1}xdx = \frac{\pi}{4} - \frac{1}{2}\log 2$

6. Find the reduction formula for $\int_0^{\frac{\pi}{2}} sin^n x dx$

Solution:

$\int_0^{\frac{\pi}{2}} sin^n x dx$ ……(1)

(1) ⇒ $I_n = \left[\frac{-1}{n} cosx sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} sin^{n-2} x dx$

$= 0 - 0 + \frac{n-1}{n} I_{n-2}$

i.e., $I_n = \frac{n-1}{n} I_{n-2}$

$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$

$$I_n = \begin{cases} \frac{n-1}{n} \quad \frac{n-3}{n-2} \quad \frac{n-5}{n-4} \cdots \frac{1}{2} I_0 (if\ n\ is\ even) \\ \frac{n-1}{n} \quad \frac{n-3}{n-2} \quad \frac{n-5}{n-4} \cdots \frac{2}{3} I_1 (if\ n\ is\ odd) \end{cases}$$

7. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cosec^3 x dx$

Solution:

Let I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cosec^3 x dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cosec^2 x cosecx dx$

Put $u = cosecx$

$du = -cosecx \cot x\, dx$

$dv = cosec^2 x dx$

$v = \int cosec^2 x dx = -cotx$

$\int u\, dv = uv - \ vdu$

∴ I = $[cosecx\ (-cotx)]_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (-cotx)(-cosecx) cotx dx$

$= \left(\frac{-2}{\sqrt{3}} \frac{1}{\sqrt{3}}\right) - \left(-2\sqrt{3}\right) - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cosecx cot^2 x dx$

$$= -\frac{2}{3} + 2\sqrt{3} - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cosecx(cosec^2x - 1)dx$$

$$\text{I} = -\frac{2}{3} + 2\sqrt{3} - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cosec^3xdx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cosecx\,dx$$

$$= -\frac{2}{3} + 2\sqrt{3} - I + [\log(cosecx - cotx]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$2\text{I} = -\frac{2}{3} + 2\sqrt{3} - \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) - \log(2 - \sqrt{3})$$

$$= -\frac{2}{3} + 2\sqrt{3} - \log\left(\frac{1}{\sqrt{3}}\right) - \log(2 - \sqrt{3})$$

$$= -\frac{2}{3} + 2\sqrt{3} + \log 1 - log\sqrt{3} - \log(2 - \sqrt{3})$$

$$\text{I} = -\frac{1}{3} + \sqrt{3} - \frac{1}{2}(2\sqrt{3} - 3)$$

8. Evaluate $\int \frac{x^2}{\sqrt{(9-x^2)}}$

Solution:

Let I = $\int \frac{x^2}{\sqrt{(9-x^2)}}$

Put $x = 3sin\theta$

$dx = 3cos\theta d\theta$

$$\text{I} = \int \frac{9sin^2\theta}{\sqrt{(9-9sin^2\theta)}} 3cos\theta d\theta$$

$$= \int \frac{9sin^2\theta}{\sqrt{(3cos\theta)}} 3cos\theta d\theta$$

$$= 9\int sin^2\theta\, d\theta$$

$$= \int \frac{1-cos\theta}{2}\, d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{sin2\theta}{2}\right] + c$$

$$= \frac{9}{2}\theta - \frac{9}{4}(2sin\theta cos\theta) + c$$

$$= \frac{9}{2}\theta - \frac{9}{2}(sin\theta cos\theta) + c$$

$$= \frac{9}{2} sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\frac{\sqrt{(9-x^2)}}{3} + c$$

$$= \frac{9}{2} sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}(x)\sqrt{(9-x^2)} + c$$

9. Integrate the following w. r. t. x

$$\int \frac{2x+3}{x^2+x+1} dx$$

Solution:

Let $2x + 3 = A \, \frac{d}{dx}(x^2 + x + 1) + B$

$2x + 3 = A(2x + 1) + B$ ….(1)

Equating the coefficients of x we get

$2 = 2A$

$A = 1$

Put $x = 0$. We get

$3 = A + B$

$3 = 1 + b$

$B = 2$

Therefore (1)⇒ $2x + 3 = (2x + 1) + 2$

$$\int \frac{2x+3}{x^2+x+1} dx = \int \frac{2x+1}{x^2+x+1} dx + \int \frac{2}{x^2+x+1} dx$$

$$= \log(x^2 + x + 1) + 2\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}} dx$$

$$= \log(x^2 + x + 1) + 2\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

$$= \log(x^2 + x + 1) + 2 \, \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \log(x^2 + x + 1) + \frac{4}{(\sqrt{3})} tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

10. Evaluate $\int_{-\infty}^{\infty} xe^{-x^2}dx$

Solution:

Take $\int xe^{-x^2}\,dx$

Put $u = x^2$

$du = 2xdx$

$\int xe^{-x^2}\,dx = \int e^{-u}\frac{du}{2}$

$= \frac{1}{2}\left[\frac{e^{-u}}{-1}\right]$

$= -\frac{1}{2}e^{-u}$

$= -\frac{1}{2}e^{-x^2}$ ….(1)

$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$ ….(2)

Take $\int_{-\infty}^{0} xe^{-x^2}dx = \lim_{t\to-\infty} \int_{1}^{\infty} xe^{-x^2}dx$

$= \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_t^0$ by (1)

$= \lim_{t\to-\infty}\left[-\frac{1}{2}+\frac{1}{2}e^{-t^2}\right]$

$= -\frac{1}{2}$

Take $\int_{0}^{\infty} xe^{-x^2}dx = \lim_{t\to\infty}\int_{0}^{t} xe^{-x^2}dx$

$= \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_o^t$ by (1)

$= \lim_{t\to-\infty}\left[\frac{1}{2}-\frac{1}{2}e^{-t^2}\right]$

$= \frac{1}{2}$

(2) $\Rightarrow \int_{-\infty}^{\infty} xe^{-x^2}dx = -\frac{1}{2}+\frac{1}{2} = 0$

UNIT 5 - MULTIPLE INTEGRALS

PART A

1. Evaluate $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}}\, dydx$

Solution: Let I = $\int_0^2 \left(xe^{\frac{y}{x}}\right)_0^{x^2} dx = \int_0^2 x(e^x - 1)dx$

$$= \left[(x)(e^x - x) - (1)\left(e^x - \frac{x^2}{2}\right)\right]_0^2$$

$$= 2e^2 - 4 - e^2 + 2 + 1$$

$$= (2)(e^2 - 2) - [(e^2 - 2) - (1)]$$

$$= e^2 - 1$$

2. Evaluate: $\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$

Solution:Let I = $\int_0^a y \int_0^{\sqrt{a^2-x^2}} dx = \int_0^a \sqrt{a^2 - x^2}dx$

$$= \left[\frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_0^a$$

$$\frac{= a^2}{2} sin^{-1}(1)$$

$$= \frac{\pi a^2}{4}$$

1. Evaluate: $\int_0^{\pi/2} \int_0^{\infty} \frac{rdrd\theta}{(r^2+a^2)^2}$

Solution: *Let* $I = \int_0^{\pi/2} \left[\int_0^{\infty} \frac{rdr}{(r^2+a^2)^2}\right] d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{-1}{(r^2 + a^2)^2}\right)_0^{\infty} d\theta$$

$$= \frac{1}{2a^2}(\theta)_0^{\pi/2}$$

$$= \frac{\pi}{4a^2}$$

2. Evaluate: $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{2cos\theta} r^2 drd\theta$

Solution: Let I = $\int_{-\pi/2}^{\pi/2} \left(\frac{r^3}{3}\right)_0^{2cos\theta} d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2} (8cos^3\theta - 0)\, d\theta$

$$= \frac{8}{3} \times 2 \int_0^{\pi/2} cos^3\theta\, d\theta$$

$$= \frac{16}{3}\frac{2}{3} = \frac{32}{9}$$

1. **Find the limits of integration in the double integral** $\iint_R f(x,y)dxdy, where\ R\ is\ in\ the\ first\ quadrant\ and\ bounded\ by: x = 0, y = 0, x + y = 1.$

Solution: The limits are: y varies from 0 to 1 and x varies from 0 to 1-y.

2. **Change the order of integration** $\int_0^a \int_y^a f(x,y)dxdy$

Solution: The given region of integration is bounded by y=0, y=a, x=y & x=a.

After changing the order, we have, I = $\int_0^a \int_0^x f(x,y)dydx$

3. **Change the order of integration for the double integral** $\int_0^1 \int_0^x f(x,y)dxdy$

Solution: $\int_0^1 \int_y^1 f(x,y)dxdy$

1. **Find the smaller area bounded by y = 2-x and $x^2+y^2=4$.**

Solution: Required area = $\int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dydx \int_0^2 [\sqrt{4-x^2} - (2-x)]\, dx$

$$= \left[\frac{x}{2}\sqrt{4^2 - x^2} + 2sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2 = \pi - 2$$

1. **Evaluate:** $\int_0^a \int_0^b \int_0^c (x + y + z)\, dzdydx$

Solution: Let I = $\int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2}\right)_0^c dydx$

$$= \int_0^a \int_0^b \left(cz + cy + \frac{c^2}{2}\right)\ dydx$$

$$= \int_0^a \left(cxy + \frac{cy^2}{2} + \frac{c^2y}{2}\right)_0^b dx$$

$$= \int_0^a \left(bcx + \frac{cb^2}{2} + \frac{c^2b}{2}\right)\ dx$$

$$= \left(\frac{bcx^2}{2} + \frac{cb^2x}{2} + \frac{c^2bx}{2}\right)_0^a$$

$$= \frac{bca^2}{2} + \frac{cb^2a}{2} + \frac{c^2ba}{2}$$

$$= \frac{abc}{2}(a + b + c)$$

2. Evaluate: $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdxdy$

Solution: Let I= $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dxdy$

$$= \int_0^a \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} + \left(\frac{(a^2-y^2)}{2}\right) sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{\pi}{4}\int_0^a (a^2-y^2)\, dy = \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi a^3}{6}$$

3. Evaluate: $\int_0^1 \int_0^{\sqrt{1^2-x^2}} \int_0^{\sqrt{1^2-x^2-y^2}} xyzdzdydx$

Solution: Let I=$\int_0^1 \int_0^{\sqrt{1^2-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1^2-x^2-y^2}} dzdydx$

$$= \frac{1}{2}\int_0^1 x\left[\int_0^{\sqrt{1^2-x^2}} y(1-x^2-y^2)dy\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1^2-x^2)^2}{2} - \frac{(1^2-x^2)^2}{4}\right]dx = \frac{1}{8}\int_0^1 x\,(1^2-x^2)^2 dx$$

$$= \frac{-1}{16}\int_0^1 (1^2-x^2)^2(-2xdx) = \frac{-1}{16}\left[\frac{(1^2-x^2)^3}{3}\right]_0^1 = \frac{1}{48}$$

4. Evaluate: $\int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 sin\varphi drd\varphi d\theta$

Solution: Let I =$\int_0^{2\pi} d\theta \int_0^{\pi} \left(\frac{r^5}{5}\right)_0^a sin\varphi d\varphi$

$$= \frac{a^5}{5}\int_0^{2\pi} d\theta \int_0^{\pi} sin\varphi d\varphi = \frac{a^5}{5}\int_0^{2\pi} (-cos\varphi)_0^{\pi}\, d\theta = \frac{2a^5}{5}\int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}$$

PART B

PROBLEMS BASED ON DOUBLE INTEGRATION IN CARTESIAN COORDINATES

1. **Evaluate: $\iint xy(x+y)\,dxdy$ over the area between y = x^2 and y = x.**

   Solution: The limits are: x varies from 0 to 1 and y varies from x^2 to x.

$$I = \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\,dydx$$

$$= \int_0^1 \left(x^2\left(\frac{y^2}{2}\right)_{x^2}^{x} + x\left(\frac{y^3}{3}\right)_{x^2}^{x} \right) dx$$

$$= \int_0^1 x\left\{\left(\frac{x^3}{2}+\frac{x^3}{3}\right)-\left(\frac{x^5}{2}+\frac{x^6}{3}\right)\right\} dx$$

$$= \left(\frac{x^5}{6}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$= \frac{3}{56}$$

2. **Evaluate: $\iint x^2y^2dxdy$ over the region in the first quadrant of the circle $x^2+y^2=1$.**

   Solution: In the given region, y varies from 0 to $\sqrt{1-x^2}$ and x varies from 0 to 1.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^2y^2\,dydx$$

$$= \int_0^1 x^2\left(\frac{y^3}{3}\right)_0^{\sqrt{1-x^2}} dx = \frac{1}{3}\int_0^1 x^2(1-x^2)^{3/2}\,dx$$

Put x = sinθ. Then dx = cosθdθ. θ varies from 0 to π/2.

$$\therefore I = \frac{1}{3}\int^{\pi/2} sin^2\theta cos^4\theta\,d\theta = \frac{1}{3}\int^{\pi/2}(1-cos^2\theta)\,cos^4\theta d\theta = \frac{1}{3}\left[\int^{\pi/2} cos^4\theta d\theta - \int^{\pi/2} cos^6\theta d\theta\right]$$

3. **Evaluate:** $\int_0^1\left[\int_x^{\sqrt{x}}(x^2y+xy^2)dydx\right]$

Solution : Let $I = \int_0^1\left[\int_x^{\sqrt{x}}(x^2y+xy^2)dydx\right] = \int_0^1\left[\int_x^{\sqrt{x}}(x^2y+xy^2)dydx\right] = \int_0^1\left(\frac{x^2y^2}{2}+\frac{xy^3}{3}\right)_x^{\sqrt{x}} dx$

$$= \int_0^1\left[\left(\frac{x^3}{2}+\frac{x.xy^{3/2}}{3}\right)-\left(\frac{x^4}{2}+\frac{x^4}{3}\right)\right]dx = \left[\frac{x^4}{8}+\frac{x^{7/2}}{(7/2)(3)}-\frac{5}{6}\left(\frac{x5}{5}\right)\right]_0^1$$

$$= \left(\frac{1}{8}+\frac{2}{21}+\frac{1}{6}\right)-(0) = \left(\frac{21+16-28}{168}\right) = \frac{9}{168} = \frac{3}{56}$$

PROBLEMS BASED ON POLAR FORMS USING DOUBLE INTEGRATION

1. **Evaluate: $\iint r^2 sin\theta dr d\theta$ over the cardioids r = a (1+cosθ).**

Solution: The limits of r: 0 to a (1+cosθ) and The limits of θ: 0 to π.

$$I = \int_0^\pi \int_0^{a(1+\cos\theta)} r^2 sin\theta\, dr d\theta = \int_0^\pi \left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)} sin\theta d\theta$$

$$= \frac{a^3}{3}\int_0^\pi sin\theta(1+cos\theta)^3 d\theta$$

Put 1+cosθ = t then –sinθ dθ = dt

When θ = 0, t = 2

When θ = π, t = 0.

$$\therefore I = \frac{a^3}{3}\int_0^2 t^3\, dt = \frac{a^3}{3}\left(\frac{a^4}{4}\right)_0^2 = \frac{4a^3}{3}$$

2. **Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ dxdy$ using polar cordinates**

Solution: $x = rcos\theta, y = rsin\theta, dxdy = rdrd\theta$ are the polar coordinates for the above integral

$$I = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2} r dr d\theta = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\ \frac{1}{2}\, d(r^2) d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[-e^{-r^2}\right]_0^\infty d\theta$$

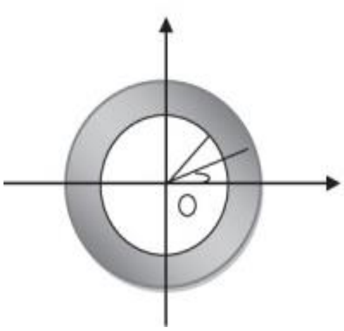
$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[e^{-r^2}\right]_0^\infty d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}}[0-1]d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4}$$

3. **By Transforming into polar coordinates , Evaluate $\iint \left(\frac{x^2y^2}{x^2+y^2}\right) dxdy$ over annular region between the circles $x^2+y^2=10$ and $x^2+y^2=4$**

Solution: Putting $x = rcos\theta,\ y = rsin\theta \Rightarrow r^2cos^2\theta + r^2sin^2\theta = 10 \Rightarrow r^2 = 10 \Rightarrow r = \sqrt{10}$

$x = rcos\theta, \quad y = rsin\theta \Rightarrow r^2cos^2\theta + r^2sin^2\theta = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$

$and\ \theta\ varies\ from\ 0\ to\ 2\pi$



$$\iint\left(\frac{x^2y^2}{x^2+y^2}\right)dxdy=\int_0^{2\pi}\int_{\sqrt{10}}^{2}\frac{(r^2cos^2\theta)(r^2sin^2\theta)}{r^2cos^2\theta+r^2sin^2\theta}\ r\ dr\ d\theta$$

$$=\int_0^{2\pi}\int_{\sqrt{10}}^{2}r^3cos^2\theta sin^2\theta\ \ dr\ d\theta$$

$$=\int_0^{2\pi}cos^2\theta sin^2\theta\left(\frac{r^4}{4}\right)_{\sqrt{10}}^{2}\ d\theta$$

$$=\int_0^{2\pi}cos^2\theta sin^2\theta\left(\frac{(2^4-\left(\sqrt{10}\right)^4}{4}\right)\ d\theta$$

$$=\frac{16-100}{4}\int_0^{2\pi}cos^2\theta sin^2\theta\ \ d\theta$$

$$=(-21)(4)\int_0^{\frac{\pi}{2}}cos^2\theta sin^2\theta\ \ d\theta$$

$$=(-84).\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2}=-\frac{84\pi}{16}=-\frac{21\pi}{4}1$$

PROBLEMS ON CHANGE THE ORDER OF INTEGRATION

1. Change the order of integration in I = $\int_0^1\int_{x^2}^{2-x}xy\,dydx$ and hence evaluate it.

Solution: Let I = $\int_0^1\int_{x^2}^{2-x}xy\,dydx$

The given region of integration is bounded by x=0, x=1, y=x^2 and x+y=2.

In the given integration x is fixed and y is varying.

So, after changing the order we have to keep y fixed and x should vary.

After changing the order we've two regions R_1 & R_2

I = I_1+I_2

I = $\int_0^1\int_0^{\sqrt{y}}xy\,dxdy+\int_1^2\int_1^{2-x}xy\,dxdy$

$$=\frac{1}{2}\left[\int_0^1(x^2y)_0^{\sqrt{y}}\,dy\int_1^2(x^2y)_0^{2-y}\,dy\right]$$

$=\frac{1}{2}\left[\int_0^1 y^2\,dy+\int_1^2 y(2-y)^2\,dy\right]$

$$=\frac{1}{2}\left[\left(\frac{y^3}{3}\right)_0^1+\left(2y^2-\frac{4}{3}y^3+\frac{y^4}{4}\right)_1^2\right]=\frac{1}{6}+\frac{5}{24}=\frac{3}{8}$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$ by changing the order of integration.

Solution: The given region is bounded by x=0, x=1, y=x and $x^2+y^2=2$.

I = $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$

After changing the order we've,

The region R is splinted into two regions R_1& R_2.

In R_1: limits of x: 0 to y & limits of y: 0 to 1

In R_2: limits of x: 0 to $\sqrt{2-y^2}$ & limits of y: 1 to $\sqrt{2}$

I = I_1+I_2

$I_1 = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}} dxdy$

$$= \int_0^1 \left(\sqrt{x^2+y^2}\right)_0^y dy = (\sqrt{2}-1)\int_0^1 y\, dy = (\sqrt{2}-1)\left(\frac{1}{2}\right)$$

$I_2 = \int_0^{\sqrt{2}} \int_x^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} dxdy$

$$= \int_1^{\sqrt{2}} \left(\sqrt{x^2+y^2}\right)_0^{\sqrt{2-y^2}} dy = \int_1^{\sqrt{2}} (\sqrt{2}-y)dy = (\sqrt{2}-1)\left(\frac{y2}{2}\right)_0^1 + \sqrt{2}(y)_1^{\sqrt{2}} - \left(\frac{y2}{2}\right)_1^{\sqrt{2}}$$

$$= (2-\sqrt{2}) - \frac{1}{2}$$

I = $(\sqrt{2}-1)\left(\frac{1}{2}\right) + (2-\sqrt{2}) - \frac{1}{2}$

$$= 1 - \frac{1}{\sqrt{2}}$$

3. Evaluate by changing the order of integration in $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dydx$

Solution: Let I = $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dydx$

The given region of integration is bounded by x=0, x=4, $y = x^2/4$, $y^2 = 4x$

After changing the order we've

Limits of x: $y^2/4$ to $2\sqrt{y}$

Limits of y: 0 to 4

I = $\int_0^4 \int_{y^2/4}^{2\sqrt{y}} dxdy$= 16/3.

4. Change the order of integration in $I=\int_0^1\int_{x^2}^{2-x} f(x,y)dydx$

Solution: Given $I=\int_0^1\int_{x^2}^{2-x} f(x,y)dydx$

The given region of integration is bounded by x=0, x=1, $y=x^2$ and x+y=2

In the given integration x is fixed and y is varying

So, after changing the order we have to keep y fixed and x should vary.

After changing the order we have two regions R_1 & R_2

$I = I_1 + I_2$

$$I=\int_0^1\int_0^{\sqrt{y}} f(x,y)dxdy+\int_1^2\int_1^{2-x} f(x,y)dxdy$$

PROBLEMS BASED ON AREA AS DOUBLE INTEGRAL

1. **Find the area of the region outside the inner circle r=2cosθ and inside the outer circle r=4 cosθ by double integration.**

Solution: Required Area = $\iint r dr d\theta$

$$=2\int_0^{\pi/2}\int_{2cos\theta}^{4cos\theta} r dr d\theta$$

$$=\int_0^{\pi/2}(r^2)_{2cos\theta}^{4cos\theta}\, d\theta = 12\int_0^{\pi/2} cos^2\theta = 12\times\frac{1}{2}\times\frac{\pi}{2} = 3\pi sq.units$$

2. Find the area of the circle of radius 'a' by double integration.

Solution: Transforming Cartesian in Polar coordinates

(i.e.) x=rcosθ & y=rsinθ. Then dxdy = rdrdθ

limits of θ: 0 to$\frac{\pi}{2}$ and limits of r: o to $2a$cosθ

Required Area = 2xupper area

$$=2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r dr d\theta$$

$$=\int_0^{\frac{\pi}{2}}(r^2)_0^{2a\cos\theta}\, d\theta$$

$$=4a^2\int_0^{\frac{\pi}{2}} cos^2\theta\, d\theta = 4a^2\frac{1}{2}.\frac{\pi}{2} = \pi a^2 sq.units$$

3. Find $\iint r^3\,drd\theta$ over the area bounded between the circles r = 2sinθ & r= 4sinθ.

Solution: In the region of integration, r varies from r=2sinθ& r= 4sinθ and θ varies from0to π.

$$I = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,drd\theta$$

$$= \frac{1}{4}\int_0^{\pi}(r^4)_{2\sin\theta}^{4\sin\theta}\,d\theta = \frac{24}{4}\int_0^{\pi} sin^4\theta\,d\theta = 60\times 2\int_0^{\frac{\pi}{2}} sin^4\theta\,d\theta$$

$$= 120\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{45\pi}{2} sq.units.$$

4. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution: Area of the ellipse = 4 x area of the first quadrant $=4\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$

$$= 4\int_0^a (y)_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\frac{x}{a}\right]_0^a = \frac{4b}{a}\left(\frac{a^2}{2}\right)\frac{\pi}{2} = \pi ab sq.units$$

5. Find the area inside the circle r=asinθ but lying outside the cardiod r=a(1-cosθ)

Solution: Given curves are r=asinθ and r =a(1-cosθ)

The curves intersect where a sin θ = a (1-cosθ)

$\Rightarrow$ a sin θ = a –a cosθ $\quad\Rightarrow$ a sin θ + a cosθ = a $\quad\Rightarrow$ sin θ + cosθ =1

$$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\frac{1}{\sqrt{2}} \qquad \Rightarrow \sin\theta\cos\frac{\pi}{4}+\cos\theta\frac{\pi}{4}\cos\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin(\theta+\frac{\pi}{4})=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4} \qquad \Rightarrow \theta+\frac{\pi}{4}=\frac{\pi}{4}(or)\pi-\frac{\pi}{4}$$

$$\Rightarrow \theta=0(or)\theta+\frac{\pi}{4}=\pi-\frac{\pi}{4} \qquad \Rightarrow \theta=\pi-\frac{2\pi}{4}=\pi-\frac{\pi}{2}=\frac{\pi}{2}$$

$$\Rightarrow \theta=0(or)\theta=\frac{\pi}{2}$$

$$\therefore \text{The required area} = \int_0^{\pi/4}\int_{a(1-\cos\theta)}^{a\sin\theta} r dr d\theta = \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta-(1+\cos^2\theta-2\cos\theta)\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta-\cos^2\theta-1+2\cos\theta\right)d\theta = \frac{a^2}{2}\left[\int_0^{\pi/2}\left(1-\cos^2\theta-\cos^2\theta-1+2\cos\theta\right)d\theta\right]$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(2\cos\theta-2\cos^2\theta\right)d\theta \qquad = \frac{a^2}{2}.2\int_0^{\pi/2}\left(\cos\theta-\cos^2\theta\right)d\theta$$

$$= a^2\left[\left(\sin\theta\right)_0^{\pi/2}-\int_0^{\pi/2}\cos^2\theta d\theta\right] \qquad = a^2\left[1-\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\right]$$

$$= a^2\left[1-\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right] \qquad = a^2\left[1-\frac{1}{2}\left(\frac{\pi}{2}+0\right)-0\right]$$

$$= a^2\left[1-\frac{\pi}{4}\right] = \frac{a^2(4-\pi)}{4}$$

6.Find by double integration, the area enclosed by the curves $y^2 = 4ax$ *and* $x^2 = 4ay$

Ans: $y^2 = 4ax \ldots\ldots\ldots\ldots\ldots (1) \qquad x^2 = 4ay \ldots\ldots\ldots\ldots (2)$

Sub (1) in (2) we get $\left[\frac{y^2}{4a}\right]^2 = 4ay$

$$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 = 64a^3y$$

$$y^4 - 64a^3y = 0 \Rightarrow y(y^3-64a^3) = 0$$

$$y = 0 \ or \ y^3 - 64a^3 = 0 \Rightarrow y^3 = 64a^3 \Rightarrow y = \sqrt[3]{64a^3} \Rightarrow y = 4a$$

$$If \ y = 0 \Rightarrow x = 0; \quad y = 4a \Rightarrow x = 4a$$

Therefore the point of intersection of (1)&(2) is (0,0) and (4a,4a)

x Varies from 0 to 4a and **y** varies from $\frac{x^2}{4a}$ *to* $2\sqrt{ax}$

The required Area $= \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx = \int_0^{4a}[y]_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx = \int_0^{4a}[2\sqrt{ax}-\frac{x^2}{4a}]\, dx$

$$= \left[2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a} = \frac{16}{3}$$

PROBLEMS BASED ON TRIPLE INTEGRAL

1. Evaluate $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

Solution : $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_0^{\log a}\int_0^{x}[e^{x+y+z}]_0^{x+y}\,dy\,dx = \int_0^{\log a}\int_0^{x}(e^{2(x+y)} - e^{x+y})\,dy\,dx$

$$= \int_0^{\log a}\left(\frac{e^{2(x+y)}}{2} - e^{x+y}\right)_0^{x} dx = \int_0^{\log a}\left[\left(\frac{1}{2}e^{4x} - e^{2x}\right) - \left(\frac{e^{2x}}{2} - e^{x}\right)\right]dx$$

$$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^{x}\right)dx = \left[\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^{x}\right]_0^{\log a}$$

$$= \left(\frac{1}{8}e^{4\log a} - \frac{3}{4}e^{2\log a} + e^{\log a}\right) - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$= \frac{1}{8}a^4 - \frac{3}{4}a^2 + a - \frac{3}{8}$$

2. Evaluate $\int_0^{a}\int_0^{b}\int_0^{c}\left(x^2 + y^2 + z^2\right)dx\,dy\,dz$

Solution : $I = \int_0^{a}\int_0^{b}\left[\frac{x^3}{3} + y^2x + z^2x\right]_0^{c} dy\,dz \quad = \int_0^{a}\int_0^{b}\left[\frac{c^3}{3} + cy^2 + cz^2\right]dy\,dz$

$$= \int_0^{a}\left[\frac{c^3y}{3} + \frac{cy^3}{3} + cyz^2\right]_0^{b} dz \quad = \int_0^{a}\left[\frac{c^3b}{3} + \frac{cb^3}{3} + cbz^2\right]dz$$

$$= \left[\frac{c^3bz}{3} + \frac{cb^3z}{3} + \frac{cbz^3}{3}\right]_0^{a} = \frac{c^3ba}{3} + \frac{cb^3a}{3} + \frac{cba^3}{3} = \frac{abc}{3}[c^2 + b^2 + a^2]$$

3. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Solution: The limits are:

Z varies from: 0 to 4-y

X varies from: $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$

Y varies from: -2 to 2.

Required volume = $2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{0}^{4-y} dzdxdy$= $2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}(4-y)dxdy$

$$=2\int_{-2}^{2}(4-y)(x)_0^{\sqrt{4-y^2}}dy= 2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy$$

$$=2\int_{-2}^{2}4\sqrt{4-y^2}\,dy-2\int_{-2}^{2}y\sqrt{4-y^2}dy$$

$=8\int_{-2}^{2}\sqrt{4-y^2}dy-2(0)$ since $y\sqrt{4-y^2}$ is an odd function.

$$=16\int_0^2\sqrt{4-y^2}\,dy=16\left[\frac{y}{2}\sqrt{4-y^2}+\frac{4}{2}sin^{-1}\frac{x}{2}\right]_0^2$$

$$=16\text{x}2\text{x}\frac{\pi}{2}=16\pi$$

4. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate plane.

Solution: The limits are:

X varies from 0 to a

Y varies from 0 to $b\left(1-\frac{x}{a}\right)$

Z varies from o to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Required Volume = $\iiint dzdydx$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)}dzdydx$$

$$=c\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\left[\left(1-\frac{x}{a}\right)-\frac{y}{b}\right]dydx$$

$$=c\int_0^a\left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)}dx$$

$$=c\int_0^a\left[b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$=\frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2dx=\frac{abc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2\left(\frac{dx}{a}\right)$$

$$=\frac{abc}{2}\left[\frac{\left(1-\frac{x}{a}\right)}{-3}\right]_0^a=\frac{abc}{6}$$

4. Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integral.

Solution: Required Volume = 8 x volume in the positive octant = $8\iiint dxdydz$

Limits of integration are:

Z varies from 0 to $\sqrt{a^2-x^2-y^2}$

Y varies from 0 to $\sqrt{a^2-x^2}$

X varies from 0 to a

$$\text{Volume} = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dzdydx \quad = 8\int_0^a\int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dydx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left(\frac{a^2-x^2}{2}\right)\sin^{-1}(1)dx = 2\pi\int_0^a a^2 - x^2\, dx = 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[\frac{2a^3}{3}\right] \quad = \frac{4\pi a^3}{3}\text{cu.units}$$

5. Find the Volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c} 1$

Solution: Required Volume = 8 x Volume in the first octant

Limits of Integration are:

Z varies from 0 to $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

Y varies from 0 to $b\sqrt{1 - \frac{x^2}{a^2}}$

X varies from 0 to a.

$$\text{Volume} = 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} (z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dydx$$

$$= \frac{8c}{b}\int_0^a\int_0^{\alpha} \sqrt{\alpha^2 - y^2}\, dydx \text{ where } \alpha = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{\alpha^2 - y^2} + \frac{\alpha^2}{2}\sin^{-1}\frac{y}{\alpha}\right]_0^a dx$$

$$= \frac{8c}{b}\frac{\pi}{4}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi bc\left[a - \frac{a}{3}\right]$$

$$= \frac{4\pi abc}{3}\text{cu. units}$$