Unit I

LOGIC AND PROOFS

PART – A

Without constructing the truth table show that $p \rightarrow (q \rightarrow p) \equiv \neg p(p \rightarrow q)$

Solution

$$p \rightarrow (q \rightarrow p) \equiv p \rightarrow (\neg q \vee p)$$
$$\equiv \neg p \vee (\neg q \vee p)$$
$$\equiv \neg p \vee (p \vee \neg q)$$
$$\equiv (\neg p \vee p) \vee \neg q$$
$$\equiv T \vee \neg q$$
$$\equiv T.$$

2. Prove that $p \rightarrow q$ is logically prove that $(\neg p \vee q)$

Solution:

p	q	$p \rightarrow q$	$\neg p \vee \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

3. Define a tautology. With an example.

A statement that is true for all possible values of its propositional variables is called a tautology universely valid formula or a logical truth.

Example: $p \vee \neg q$ is a tautology.

4. When do you say that two compound statement proposition are equivalent.

Two compound proposition are said to be equivalent if then the have identical truth tables.

5. Give an indirect proof of the theorem if 3n+2 is odd, then n is odd.

Solution:

P: 3n+2 is odd

Q: n is odd

Hypothesis: Assume that $p \rightarrow$ q is false.

Assume that p *is true and* q is false.

i.e, n is not odd $\Rightarrow$ n is even.

Analysis: If n is even then n=2k for some integer k.

$$3n+2 = 3(2k)+2.$$
$$= 6k+2.$$
$$= 2(3k+1)$$

6. Define a universal specification.

$(x)A(x) \Rightarrow A(y)$

If a statement of the form (x)A(x) is assumed to be true , then the universal quantifier can be dropped to obtain A(y) is true for any arbitrary object y in the universe.

7.Show that {∨,∧} is not functionally complete.

Solution: $\neg p$ cannot be expressed using the connectives $\{\vee, \wedge\}$ .since no sets contribution of the statement exists $\{\vee, \wedge\}$ as input if T and the output is F.

8. Write the converse, inverse, contra positive of „If you work hard then you will be rewarded"

Solution:

p: you will be work hard.
q: you will be rewarded.
¬p: You will not be work hard.
¬ q: You will no tbe rewarded.
Converse: q→ p, If you will be rewarded then you will be work hard
Contrapositive: ¬ q→ p,if You will not be rewarded then You will not be work hard.
Inverse: ¬ p→ ¬ q, if You will not be work hard then You will no tbe rewarded.

9. let E={ -1, 0,1,2 denote a universe of discourse . If P(x ,y) : x +y =1 find the truth value of $(\forall x)(\exists y)$.

Solution: Given E={ -1, 0,1,2} P(x,y): x+y=1

$(\forall x)(\exists y)$. P(x,y) is true since 2+(-1)=1
1+0=1.

10.obtain the disjunctive normal forms of p ∧(p→ q)

Solution: let s⇔ $p \wedge (p \rightarrow q)$

$- p \wedge (\neg p \vee \vee q)$

$- (p \wedge \neg p) \vee (p \wedge q)$

11. Show that $(p \wedge q) \Rightarrow (p \rightarrow q)$.

Solution:

To prove: $(p \wedge q) \rightarrow (p \rightarrow q)$. is a tautology.

p	q	$p \wedge q$	p→ q	$(p \wedge q) \rightarrow (p \rightarrow q)$.
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

12.Write an equivalent formula for p∧(q ↔r) which contains neither the bi onditional nor conditional.

Solution :

$p \wedge (q \leftrightarrow r) \Longleftrightarrow (p \wedge (q \rightarrow r) \wedge (r \rightarrow q)$

$- (p \wedge (\neg q \vee r) \wedge (\neg r \vee q).$

13. Show that (x) (H(x) →M(x)) ∧ H(s) ⇒ M(s)

Solution :

Steps	Premises	Rule	Reason
1	(x) (H(x) →M(x))	P	Given premise
2	H(s) →M(s)	US (1)	$(\forall x)\ p\ (x) \Rightarrow p(y)$
3	H(s)	P	Given premise
4	M(s)	T	(2) (3) $(p \rightarrow q, p \Rightarrow q)$

14. Show that ¬ p(a,b) follows logically from (x) (y) (p(x,y) →w(x,y)) *and* ¬w(a,b)

Solution :

1. (x) (y) (p(x,y) → w(x,y) p
2. (y), p(a,y) → w(a,y) US, (1)
3. P(a,b) → w(a,b) US (2)
4. $\neg w(a,b)$ p Given
5. $\neg p(a,b)$ T (3),(4), $(p \rightarrow Q) \wedge \neg Q \Rightarrow \neg p$

15. Symbolise: For every x, these exixts a y such that $x^2+y^2 \geq 100$

Solution :

$(\forall x)(\exists y)(x^2+y^2 \geq 100)$

PART – B

1. a) Prove that $(P \to Q) \wedge (Q \to R) \to (P \to R)$

Proof:

Let S: $(P \to Q) \wedge (Q \to R) \to (P \to R)$

To prove: S is a tautology

P	Q	R	$(P \to Q)$	$(Q \to R)$	$(P \to R)$	$(P \to Q) \wedge (Q \to R)$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The last column shows that S is a tautology

2. Show that $\neg(p \leftrightarrow q) \equiv (p \vee q) \wedge \neg(p \wedge q)$ without constructing the truth table

Solution :

$\neg(p \leftrightarrow q) \equiv (p \vee q) \wedge \neg(p \wedge q)$

$\neg(p \leftrightarrow q) \equiv \neg(p \to q) \wedge (q \to p)$

$\equiv \neg(\neg p \vee q) \wedge (\neg q \vee p)$

$\equiv \neg(\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p)$

$\equiv \neg(\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee ((\neg p \wedge p) \vee (q \wedge p)$

$\equiv \neg(\neg p \vee q) \vee F \vee F \vee (q \wedge p)$

$\equiv \neg(\neg p \vee q) \vee (q \wedge p)$

$\equiv (p \vee q) \wedge (q \wedge p)$.

3. Obtain PCNF of $(\neg p \rightarrow r) \wedge (q \leftrightarrow p)$. and hence obtain its PDNF.

Solution:

PCNF:

$S \Longleftrightarrow (\neg p \rightarrow r) \wedge (q \leftrightarrow p)$.

– $(\neg p \rightarrow r) \wedge ((q \rightarrow p). \wedge (p \rightarrow q)$

- $(p \vee r) \wedge ((\neg q \vee p). \wedge (\neg p \vee q)$

- $((p \vee r) \vee F) \wedge ((\neg q \vee p). \vee F) \wedge ((\neg p \vee q) \vee F)$

- $((p \vee r) \vee (q \wedge \neg q)) \wedge ((\neg q \vee p). \vee (r \wedge \neg r)) \wedge ((\neg p \vee q) \vee (p \wedge \neg p))$.

- $((p \vee r \vee q) \wedge (p \vee r \vee \neg q)) \wedge ((\neg q \vee p \vee r) \wedge .(\neg q \vee p \vee \neg r) \wedge ((\neg p \vee q \vee r) \vee (\neg p \vee q \vee \neg r)$

– $((p \vee r \vee q) \wedge ((\neg q \vee p \vee r) \wedge .(\neg q \vee p \vee \neg r) \wedge ((\neg p \vee q \vee r) \vee (\neg p \vee q \vee \neg r)$

PCNF of S: $((p \vee r \vee q) \wedge ((\neg q \vee p \vee r) \wedge .(\neg q \vee p \vee \neg r) \wedge ((\neg p \vee q \vee r) \vee (\neg p \vee q \vee \neg r)$

PCNF of $\neg$S: $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r| \quad) \wedge (\neg p \vee \neg q \vee \neg r)$

PDNF of S: $(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \vee \wedge \neg q \wedge \neg r)$.

4. Prove that $\sqrt{2}$ is irrational.

Solution :

Suppose $\sqrt{2}$ is irrational.

$\therefore \sqrt{2} = \frac{p}{q}$ for p,q $\in z, q \neq 0, p \& q$ have no common divisor.

$\therefore \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2$.

Since p^2 is an even integer, p is an even integer.

$\therefore$ p= 2m for some integer m.

$\therefore\ (2m)^2 = 2q^2 \Rightarrow q^2 = 2m^2$

Since q^2 is an even integer, q is an even integer.

$\therefore$ q= 2k f or some integer k.

Thus p & q are even . Hence they have a common factor 2. Which is a contradiction to our assumption.

$\therefore \sqrt{2}$ is irrational.

5. Verify that validating of the following inference. If one person is more successful than another, then he has worked harder to deserve success. Ram has not worked harder than Siva. Therefore, Ram is not more successful than Siva.

Solution:

Let the universe consists of all persons.

Let S(x,y): x is more successful than y.

H(x,y): x has worked harder than y to deserve success.

a: Ram

b: Siva

Then, given set of premises are

1) (x) (y) [S(x,y) $\rightarrow$H(x,y)]
2) $\neg$H(a,b)
3) Conslution is $\neg$S(a,b).

{1}	1) (x) (y) [S(x,y) $\rightarrow$ H(x,y)]	Rule P
{2}	2) (y) [S(a,y) $\rightarrow$ H(a,y)]	Rule US
{3}	3) [S(a,b) $\rightarrow$ H(a,b)]	Rule US
{4}	4) $\neg$H(a,b)	Rule P
{5}	5) $\neg$ S(a,b)	Rule T ($\neg$P, P $\rightarrow$Q $\Rightarrow$ $\neg$Q)

Unit – II

COMBINATORICS

PART - A

1. Pigeon Hole Principle:

If (n=1) pigeon occupies 'n' holes then atleast one hole has more than 1 pigeon.
Proof:
Assume (n+1) pigeon occupies 'n' holes.
Claim: Atleast one hole has more than one pigeon.
Suppose not, ie. Atleast one hole has not more than one pigeon.
Therefore, each and every hole has exactly one pigeon.
Since, there are 'n' holes, which implies, we have totally 'n' pigeon.
Which is a $\Rightarrow\Leftarrow$ to our assumption that there are (n+1) pigeon.
Therefore, atleast one hole has more than 1 pigeon.

2. Prove that $nP_r = (n-r+1)*nP_{r-1}$

Solution:

We know that $nP_r = \dfrac{n!}{(n-r)!}$

$$nP_{r-1} = \frac{n!}{[n-(r-1)]!}$$

But n! = n*(n-1)!

(n-r+1)! = (n-r+1) (n-r)!

Now (n-r+1) *nP_{r-1}

$$= (n-r+1) * \frac{n!}{[n-(r-1)]!}$$

$$= \frac{(n-r+1)*n!}{[n-r+1)!} = \frac{n!}{(n-r)!} = nP_r$$

3. In how many ways can letters of the word "INDIA" be arranged?

Solution:

The word contains 5 letters of which 2 are I's.

The number of words possible = $\frac{5!}{2!}$ = $\frac{5*4*3*2*1}{2*1}$ = 60

4. Use mathematical induction to show that $n! \geq 2^{n+1}$, n = 5,6,…..

Solution:

Let P(n): $n! \geq 2^{n+1}$, n = 5,6,…..

Assume P(5): $5! \geq 2^{5+1}$ is true

Assume P(k): $k! \geq 2^{k+1}$ is true

Claim: P(k+1) is true.

Using (1), we have, $k! \geq 2^{k+1}$

Multiply both sides by 2, we have $2\,k! \geq 2.\,2^{k+1}$ is true

$(k+1)\,k! \geq 2^{k+2}$

$(k+1)\,! \geq 2^{k+2}$

$\therefore$ P(k+1) is true

Hence by the principle of mathematical induction, $n! \geq 2^{n+1}$, for n = 5,6,…..

5. Find the value of n if $nP_3 = 5nP_2$

Solution:

$nP_3 = 5nP_2$

$n(n-1)\,(n-2) = 5\,n\,(n-1)$

$n-2 = 5$

$n=7$

6. How many ways are these to select five players from 10 member tennis team to make a trip to match to another school.

Solution:

5 members can be selected from 10 members in $10C_5$ ways.

Now, $10C_5 = \frac{10!}{5!5!}$ = 252 ways.

7. If the sequence $a_n = 3.2^n$, $n \geq 1$, then find the corresponding recurrence relation.

Solution:

For $n \geq 1$ $\qquad a_n = 3.2^n$

Now, $\qquad a_{n-1} = 3.2^{n-1} \quad = 3.\ 2^{n-1} \quad = 3.\ \frac{2^n}{2}$

$a_{n-1} = \frac{a^n}{2} \qquad \Rightarrow a_n = 2\,(a_{n-1})$

$\Rightarrow a_n = 2\,a_{n-1}$, for $n \geq 1$, with $a_0 = 3$.

PART –B

1. Find the number of distinct permutation that can be formed. From all the letter of each word (1) RADAR (2) UNUSAL.

Solution :

(1) The word RADAR contains 5 letters of which 2 A's and 2 R's are there

The number of possible words = $\frac{5!}{2!2!}$

$= \frac{120}{2*2} = 30$

No. of distinct permutation = 30

The word UNUSAL. contains 7 letters of which 3 U's are there

(2) The number of possible words = $\frac{7!}{3!}$

$=840$

No. of distinct permutation = 840.

2. Use mathematical Induction, prove that Solution:

$\frac{1}{\sqrt{1}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{3}}$ ……… $\frac{1}{\sqrt{n}}$ $\sqrt{n}$ **+ + ….+ >for $n \geq 2$.**

Let P(n): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n}$, n ≥ 2.

i) P(2): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}=(1.707)>\sqrt{2}=(1.414)$ is true.

ii) P(k): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots\ldots+\frac{1}{\sqrt{k}}>\sqrt{k}$ is true............................ 1

Claim: P(k+1): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$

$$=\sqrt{k}+\frac{1}{\sqrt{k+1}}=\frac{\sqrt{k.\sqrt{k+1}}+1}{\sqrt{k+1}}$$

$$=\sqrt{k+1}$$

P(k+1) > $\sqrt{k+1}$ is true.

By the principle of mathematical Induction, $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n}+1$.

3. Use mathematical Induction, prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let P(n): $1^2+2^2+3^2+\ldots\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

i) P(1): $1^2=\frac{1(1+1)(2.1+1)}{6}$ is true.

ii) Assume P(k): $1^2+2^2+3^2+\ldots\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$ is true.

Where k is any integer,

iii) Claim: P(k+1) is true.

Now, P(k+1): $1^2+2^2+3^2+\ldots\ldots+k^2+(k+1)^2$

$$=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$ P(k+1) is true.

By the principle of mathematical Induction, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all 'n'

4. Use mathematical Induction, prove that (3^n+7^n-2) is divisible by 8, for $n \geq 1$.

Solution:

Let P(n): (3^n+7^n-2) is divisible by 8

i) P(1): (3^1+7^1-2) is divisible by 8, is true.

ii) Assume P(k): (3^k+7^k-2) is divisible by 8, is true 1

Claim: P(k+1) is true.

$P(k+1) = 3^{k+1}+7^{k+1}-2 \quad = 3.\ 3^k+7.7^k-2 \quad = 3.3^k+3.7^k+4-7^k-6+4$

$= 3\ (3^k+7^k-2) + 4(7^k+1)$ 2

$4(7^k+1)$ is divisible by 8. (7^k+1 is an even number, for $k \geq 1$)

$3(3^k+7^k-2)$ is divisible by 8 Using 1

$P(k+1) = 3(3^k+7^k-2) + 4(7^k+1)$ is divisible by 8 is true.

P(k+1) is true.

By the principle of mathematical Induction, P(n): (3^n+7^n-2) is divisible by 8

5. Solve the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$, $n \geq 2$ and $a_0 = 1$ & $a_1 = 2$.

Solution:

The recurrence relation can be rewritten as $a_n-2a_{n-1} + 2a_{n-2} = 0$.

The Characteristic equation is $n^2-2r+2 = 0$

Roots are $r = \frac{2 \pm 2i}{2} = 1 \pm i$

The modulus (amplitude) from of $1 \pm i = \sqrt{2}\,(\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4})$

Solution: $a_n = (\sqrt{2})^n\,(C_1.\cos\frac{n\pi}{4} + C_2.\sin\frac{n\pi}{4})$

Given $a_o = 1$, put n=0 in (A)

$a_o = (\sqrt{2})^0\,(C_1+0)$

we get $C_1 = 1$

Given $a_1 = 2$, put n=1 in (A)

$a_1 = (\sqrt{2})^1\,(C_1.\cos\frac{n\pi}{4} + C_2.\sin\frac{n\pi}{4})$

$2 = \sqrt{2}\ (C_1\ \frac{1}{\sqrt{2}} + C_2.\ \frac{1}{\sqrt{2}})$

$2 = C_1 + C_2$. $\qquad 2 = 1 + C_2$. $\qquad C_2 = 1$

Substituting $C_1 = 1$ & $C_2 = 1$ in A

The required solution is $a_n = (\sqrt{2})^n \left(\cos \frac{n\pi}{4} + i.\sin \frac{n\pi}{4}\right)$

UNIT – III

GRAPHS

PART -A

1. The Handshaking Theorem

Let G = (V, E) be an undirected graph with 'e' edges. Then $\sum_{v \in V} \deg(v) = 2e$

The sum of degrees of all the verticies of an undirected graph is twice the number of edges of the graph and hence even.

Proof:

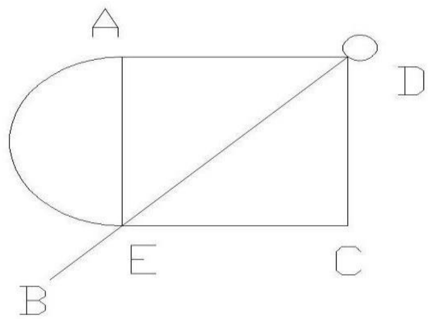
Since every edge is incident with exactly two vertices, every edges contributes 2 to the sum of the degree of the vertices.

All the 'e' edges contributes (2e) to the sum of the degrees of vertices.

$\sum \deg(v) = 2e$

2. Draw the graph with 5 vertices, A, B, C, D, E such that deg(A) = 3, is an odd vertex, deg (C) = 2 and D and E are adjacent.

Solution:



d(E) = 5

d(C) = 2

d(D) = 5

d(A) = 3

d(B) = 1

3. In an undirected graph, the numbers of odd degree vertices are even.

Proof:

Let V_1 and V_2 be the set of all vertices of even degree and set of all vertices of odd degree, respectively, in a graph G= (V, E).

Therefore, **d(v)= d(vi)+ d(vj)**

By handshaking theorem, we have

Since each deg (vi) is even, is even.

As left hand side of equation (1) is even and the first expression on the RHS of (1) is even, we have the 2nd expression on the RHS must be even.

Since each deg (vj) is odd, the number of terms contained in i.e., The number of vertices of odd degree is even.

4. If the simple graph G has 4 vertices and 5edges, then how many edges does G^c have?

Solution:

$$\left|E(G \cup G^c)\right| = \frac{\nu(\nu-1)}{2}$$

$$\left|E(G)\right| + \left|E(G^c)\right| = \frac{\nu(\nu-1)}{2}$$

$$e + \left|E(G^c)\right| = \frac{\nu(\nu-1)}{2}$$

$$\left|E(G^c)\right| = \frac{\nu(\nu-1)}{2} \qquad \therefore\ G^c \text{ has } \frac{\nu(\nu-1)}{2} - e \text{ edges}$$

$\therefore$ G^c have $\frac{4(4-1)}{2} - 5 = 6 - 5 = 1$ edges.

5. How many edges are there in a graph with ten vertices each of degree six.

Solution:

Let e be the number of edges of the graph.

2e = Sum of all degrees

$\Rightarrow$ = 10*6 = 60.

$\Rightarrow$ 2e = 60 $\Rightarrow$ e = 30. There are 30 edges.

6. How many vertices does a regular graph of degree 4 with 10 edges have.

Solution:

$$\sum d(v) = 2e$$

Let ‘n’ be the number of vertices and ‘e’ is the number of edges.

$4n = 2 * 10$

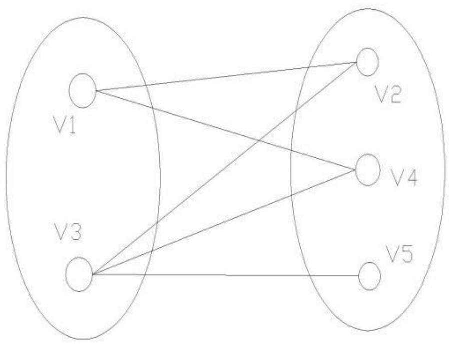
$n=5$

There are 5 vertices in a regular graph of degree 4 with 10 edges.

7. Define Bipartite Graph.

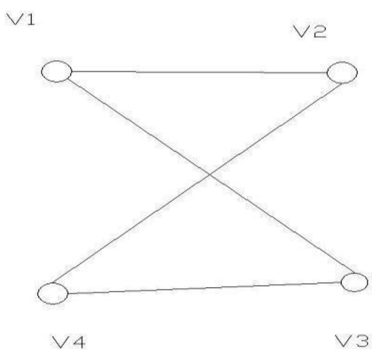
A graph G is said to be bipartite if its vertex set V(G) can be partitioned into two disjoint non empty sets V_1 and V_2, $V_1V_2=V(G)$, such that every edge in E(G) has one end vertex in V_1 and another end vertex in V_2. (So that no edges in G, connects either two vertices in V_1 or two vertices in V_2)

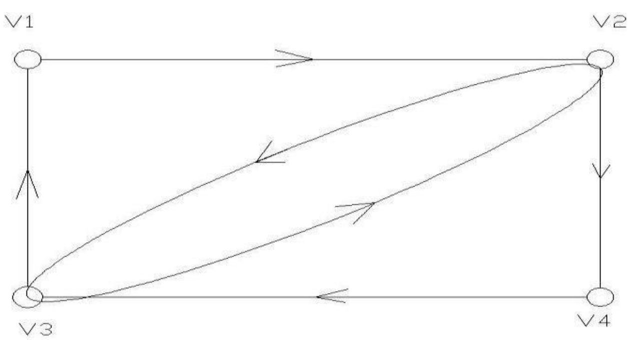
For example, consider the graph G.



Then G is a Bipartite graph.

8. Find adjacency matrix of the graphs given below





Solution:

a) Adjacency matrix

$$A = [a_{ij}] = \begin{array}{c} V1 \\ V2 \\ V3 \\ V4 \end{array} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

b)

$$A = [a_{ij}] = \begin{array}{c} V1 \\ V2 \\ V3 \\ V4 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

9. Define: Graph Isomorphism

Two graph G_1 and G_2 are said to be Isomorphic to each other, if there exist a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.

10. Define: Connect Graph

A directed graph is said to be connected if any pair of nodes are reachable from one another. That is, there is a path between any pair of modes.

A graph which is not connected is called disconnected graph.

PART – B

1. Show that graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is V_1 and the other in V_2.

Solution:

Suppose that such a partitioning exists. Consider two arbitrary vertices a & b of G such that a $\in V_1$ and b $\in V_2$. No path can exist between vertices a & b. Otherwise, there would be atleast one edge whose one end vertex be in V_1 and the other in $\in V_2$. Hence if partition exists, G is not connected.

Conversely, let G be a disconnected graph.

Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partitio.

2. A simple graph with „n" vertices and „k" components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:

Let n1,n2, ……, nk be the number of vertices in each of k components of the graph G.

Then $n_1+n_2+\ldots.+n_k = n = |V(G)|$

$$\sum_{i=1}^{k} ni = n$$

Now, $\sum_{i=1}^{k}(n_i - 1) = (n_1-1)+(n_2-1)+\ldots..+(n_k-1)$

$$= \sum_{i=1}^{k}(n_i - k)$$

$$\sum_{i=1}^{k}(n_i - 1) = n-k$$

Squaring on both sides $\left[\sum_{i=1}^{k} n_i - k\right] = (n-k)^2$

$(n_1-1)2 + (n_2-1)2 + \ldots. + (n_k -1)2 \leq n^2 + k^2 -2nk$

$n^{12}+1-2 n^1+n^{22}+1-2n^2+\ldots..+n^2k+1-2n_k \leq n^2 + k^2 -2nk$

$$\sum_{i=1}^{k} n_i^2 + k -2n \leq n^2 + k^2 -2nk$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 -2nk+2n-k$$

$$\sum_{i=1}^{k} n^2 = n^2 + k^2 -k-2nk+2n \qquad = n^2 +k(k-1)-2n(k-1)$$

i

=

n
2

$$= n^2 + (k-1) - (k-2n)$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + (k-1) - (k-2n)$$

Since, G is simple, the maximum number of edges of G in its components is $\frac{n_i(n_i-1)}{2}$.

$$\text{Maximum number of edges of G} = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \sum_{i=1}^{k} \left[\frac{(n_i^2 - n_i)}{2}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$\leq \frac{1}{2}[n^2+(k-1)(k-2n)] - \frac{n}{2}$$

$$= \frac{1}{2}[n^2 -2nk+k^2+2n-k-n)]$$

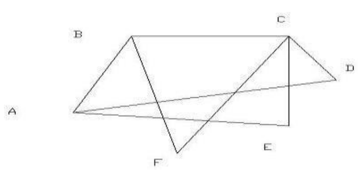
$$= \frac{1}{2}[n^2 -2nk+k^2+n-k]$$

$$= \frac{1}{2}[n-k)^2 +(n-k)]$$

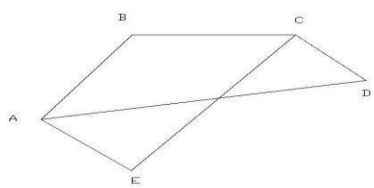
$$= = \frac{1}{2}(n-k)(n-k+1)$$

$$\text{Maximum no. of edges of G} \leq \frac{(n-k)(n-k+1)}{2}$$

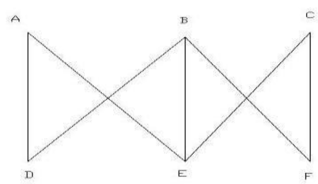
3. Determine which of the following graphs are bipartite and which are not. If a graph is bipartite, state if its completely bipartite.



G1



G2



G3

Solution:

i) In G_1, since the vertices D, E, F are not connected by edges, the may be considered as one subset V_1. Then A, B, C belong to V_2.

$V_1 = \{D,E,F\}$ & $V_2 = \{A,B,C\}$

The vertices of V_1 are connected by edges to the vertices of V_2 , but the vertices A, B, C V_2 of the subset are the edges AB, BC.

Hence the graph G_1 is not a Bipartite.

ii) By taking

$V_1 = \{A,C\}$ & $V_2 = \{B,D,E\}$

The condition required for bipartite graph are satisfied.

Hence the graph G_2 is complete Bipartite graph.

iii) By taking

$V_1 = \{A,B,C\}$ & $V_2 = \{D,E,F\}$

The condition of bipartite is satisfied.

Hence the graph G_3 is Bipartite.

Here the vertices A and F (Also C and D) are not connected by edges.

$A \in V_1$ is not adjacent to $F \in V_2$

G_3 is not a complete Bipartite graph.

4. The maximum number of edges in a simple graph with „n" vertices is $\frac{n(n-1)}{2}$

Proof:

We prove this theorem by the principle of Mathematical Induction.

For n =1, a graph with 1 vertices has no edges.

The result is true for n = 1.

For n =2, a graph with 2 vertices may have atmost one edge.

$\frac{2(2-1)}{2} = 1$

The result is true for n = 2.

Assume that the result is true for n =k ie., a graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges.

When n = k + 1, let G be a graph having 'n' vertices and G' be the graph obtained from G by deleting one vertex say $\nu \in V(G)$

Since G' has k vertices, then by the hypothesis G' has atmost $\frac{k(k-1)}{2}$ edges. Now add the vertex 'ν' may be adjacent to all the k vertices of G'.

The total number of edges in G are,

$$\frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k)}{2}$$

$$= \frac{k^2 + k}{2}$$

$$= \frac{k(k+1)}{2}$$

$$= \frac{(k+1)(k+1-1)}{2}$$

The result is true for n = k +1

Hence the maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$

5. If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is amultiple of k.

Proof:

Let 2n be the number of vertices of the given graph.

Let n_e be the number of edged of the given graph.

By Handshaking theorem, we have $\sum_{i=1}^{2n} \deg Vi = 2n_e$

$\Rightarrow 2n_k = 2n_e$ $\Rightarrow n_e = n_k$ $\Rightarrow$ no. of edges = multiple of k.

The number of edges of the given graph is a multiple of k.

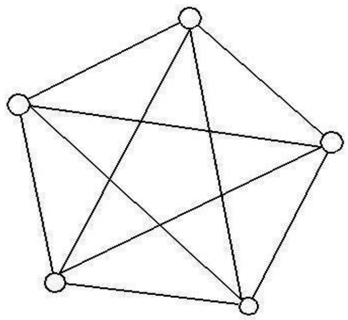
6. Draw the complete graph K_5 with vertices A, B, C, D, E. Draw all complete sub group of K_5 with 4 vertices.

Solution:

In a graph, if there exist an edge between every pair of vertices, then such a graph is called complete graph.

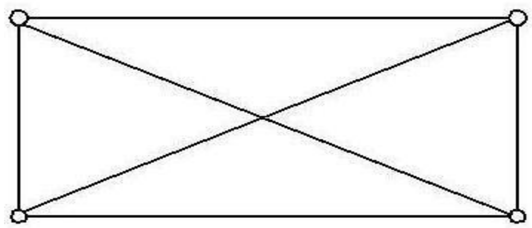
ie., In a graph if every pair of vertices are adjacent, then such a graph is called complete graph.

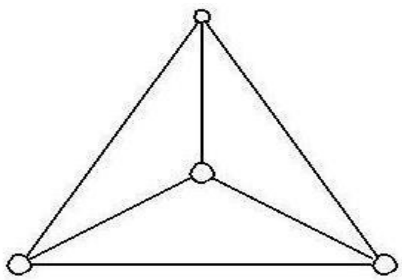
Complete graph k_5 is



k_5

Now, complete subgraph of k_5 with 4 vertices are





k_4 **k_4**

UNIT – IV
ALGEBRAIC STRUCTURES
PART - A

1. Every cyclic monoid (semi group) is commutative.

Proof:

Let (M,*) be a cyclic monoid whose generator is $a \in M$

Then for $x,y \in M$, we have $x = a^n$, $y = a^m$ m,n – integers.

Now, $x*y = a^n * a^m = a^{n+m}$

$= a^{n+m} = a^m * a^n = y * x$

Therefore (M,*) is commutative or abelian.

2. Every sub group of an abelian group is normal.

Proof:

Let G be an abelian group and H be a subgroup of G.

Now, $\chi * H * \chi^{-1} = \chi *(H*x^{-1})$, $\chi \in G$, $h \in H$ [G is abelian, $\therefore x^{-1} * H = H * x^{-1}$]

$= \chi *(x^{-1}* H)$

$= (\chi * \chi^{-1}* H)$

$= e* H$

$= H$

$\therefore$ For $\chi \in G$, $h \in H$, we have

$\chi * H * \chi^{-1} = H$

3. Every cyclic is an abelian group

Proof: $x,y \in G$

$x = a^k$, $y = a^t$ for integers k,t

$x*y = a^k*a^t = a^{k+t} = a^{t+k} = a^t*a^t = y*x$

$x*y = y*x$ $\therefore$ (G,*) is an abelian group.

4. Show that the composition of semigroup homomorphism is also a semi group homomorphism.

Proof:

$(h \circ g)\ (a*b) \quad = h\ (g(a*b)$

$= h(g(a)\ \Delta\ g\ (b)) \qquad (\therefore g \text{ is a homomorphism})$

$= h(g(a)\ \oplus h\ (g(b)) \qquad (\therefore h \text{ is a homomorphism})$

$= (h \circ g)\ (a)\ \oplus (h \circ g)\ (b)$

Hence $h \circ g$is a seni group homomorphism from $(S,*)$ to $(V, \oplus)$

5. Define Group.

Solution:

A non empty set G together with the binary operation *, ie. , (G,*) is called a group if * satisfies the following conditions.

i) Closure: $a*b \in G$, for all $a,b \in G$

ii) Associative: $(a*b) *c = a*(b*c)$, for all $a,b,c \in G$.

iii) Identity: There exists an element $e \in G$ called the identity element such that $a* e = e*a = a$, for all $a \in G$.

iv) Inverse: There exists an element $a^{-1} \in G$ called the inverse of 'a' such that $a*a^{-1} = a^{-1} * a = e$, for all $a \in G$.

6. Prove that in a group the only idempotent element is identity element.

Proof:

Let (G,*) be a group.

Assume that $a \in G$ is an idempotent element. Then we have

$a*a = a$

$a = a* e$

$= a*(a*a^{-1})$

$= (a*a)*a^{-1}$

$= a*a^{-1}$

$a = e$

a

$a = e$

ie., Idempotent element a is equal to the identity.

7. In a group (a-1)-1 = a, a $\in$G. (or) The inverse of a-1 is a.

Proof:

Let (G,*) be a group.

Let e be the identity element.

We know that

$a^{-1}*a = e = a*a^{-1}, a \in G$

$(a^{-1})^{-1}*(a^{-1}*a) = (a^{-1})^{-1}* e$

$= (a^{-1})^{-1}$

But $((a^{-1})^{-1} * a^{-1})*a = e*a$

$= a$

From (1) & (2) we get

$(a^{-1})^{-1} = a$ [Note: The above property is involution law]

8. In a group G prove that an element a $\in$G such that $a^2 = e$, a $\neq$ e if $a = a^{-1}$.

Solution:

Assume that $a = a^{-1}$

$a^2 = a*a = a*a^{-1} = e$

Conversely assume that

$a^2 = e, a \neq e$

$a*a = e$

$a^{-1}*(a*a) = a^{-1}* e$

$(a^{-1}*a)*a = a^{-1}$

$e*a = a^{-1}$

$a = a^{-1}$

9. Isomorphism

Defination:

A mapping f from a group (G,*) to a group (G', Δ) is said to be an isomorphism if

i) f is ahomomorphism. f(a*b) = f(a) Δ f(b), for all a,b $\in$G.
ii) f is one-one. (Injective)
iii) f is onto. (Surjective)

In otherwords a bijective homomorphism is said to be an isomorphism.

10. Normal subgroups.

Let H be subgroup of G under *.

Then H is said to be a normal subgroup of G, for every x $\in$G and for h $\in$H,

if $x*h*x^{-1} \in H$

$x*H*x^{-1} \subseteq H$

Alternatively, a subgroup H of G is called a normal subgroup of G if x*h=h*x for all x $\in$G.

PART – B

1. Let S =QxQ, be the set of all ordered pairs of rational numbers and given by (a,b)*(x,y)=(ax,ay+b)

i) Check (S,*) is a semigroup. Is it commutative?

ii) Also find the identity element of S.

Solution:

i) (1) Closure Property:
Obviously * satisfies closure property.
(2) Associative Property:
Consider,
[(a,b)*(x,y)]*(c,d)
= [ax,ay+b)*(c,d)]
= [axc, axd+(ay+b)]
= [axc,adx+ay+b] .... (1)

Now,

$(a,b)*[(x,y)*(c,d)]$

$= (a,b) * [xc,xd+y]$

$= [axc,a(xd+y)+b]$

$=[axc,axd+ay+b]$

$=[acx,adx+ay+b]$ .... (2)

From (1) & (2) we have

$[(a,b)*(x,y)]*(c,d)$

$= (a,b)*[(x,y)*(c,d)]$

$\therefore$ * is associative

$\therefore$ (S, *) is semigroup.

Commutative Property:

$(a,b) * (x,y) = (ax,ay+b)$ ………….. (3)

$(x,y) * (a,b) = (xa, xb + y)$

$= (ax, bx + y)$ ………….. (4)

From (3) & (4)

$(a,b) * (x,y) \neq (x,y)*(a,b)$

$\therefore$ (S, *) is not commutative.

(ii) Identity Property:

Let (e_1, e_2) be the identity element of (S,8)

Then for any $(a,b) \in S$

$(a,b)* (e_1, e_2) = (a,b)$

$(a e_1, ae_2 + b) = (a,b)$

$\Rightarrow a\ e_1=a$ and $ae_2 + b = b$

$e_1 = 1$ and $e_2 = \dfrac{b-b}{a} = 0, (a \neq 0)$

$\therefore$ The identi~~ty ele~~ment $= (e_1, e_2)$

$= (1,0)$

2. The necessary and sufficient condition that a non-empty subset H of a group G to be a subgroup is $a,b \in H \Rightarrow a*b^{-1} \in H$, for all $a,b \in H$ (closure).

Proof:

Let us assume that H is a subgroup of G. Since H itself is a group, we have for $a ,b \in H \Rightarrow a*b \in H$ (closure)

Since $b \in H$ $\Rightarrow b^{-1} \in H$ (H is a subgroup)

For $a,b \in H$ $\Rightarrow a,b^{-1} \in H$

$\Rightarrow a*b^{-1} \in H$ (H is a subgroup)

Sufficient condition:

Let $a*b^{-1} \in H$, for $a,b \in H$

Now we have to prove that H is a subgroup of G.

i) Identity: Let $a \in H$

$\Rightarrow a^{-1} \in H$

$\Rightarrow a*a^{-1} \in H$

$\Rightarrow e \in H$ Hence the identity element ‘e’$\in$H.

ii) Inverse: Let $a,e \in H$

$\Rightarrow e*a^{-1} \in H$

$\Rightarrow a^{-1} \in H$ Every element ‘a’ of H has its inverse a^{-1} is in H.

iii) Closure: Let $b \in H \Rightarrow b^{-1} \in H$

For $a,b \in H \Rightarrow a,b^{-1} \in H$

$\Rightarrow a*(b^{-1})^{-1} \in H \Rightarrow a*b \in H$ H is closed. H is a subgroup of G.

3. If * is the operation defined on s=Q*Q, the set of ordered pairs of rational numbers and given by (a,b)*(x,y)=(ax,ay+b),

a) Find if (s,*) is a semigrpoup. Is it commutative?

b) Find the identity element of S

c) Which element, if any, have inverse and what are they?

Solution:

a) {(a*b)*(x,y)*(c*d)

=(ax,ay+b)*(c,d)

$=(acx,adx+ay+b)$ Now,

$(a,b)*\{(x,y)*(c,d)\}$

$=(a,b)*(cx,dx+y)$

$(acx,adx+ay+b)$

Hence , * is associative on s.

{s,*} is a semigroup.

Now $(x,y)*(a,b) = (ax,bx+y) \neq (a,b)*(x,y)$

{s,*} is not commutative.

b) Let (e_1,e_2) be the identity element of {s,*} , Then for any $(a,b) \in S$,

$(a,b)*(e_1,e_2)=(a,b)$

Ie. $(ae_1,ae_2+b)=(a,b)$

$ae_1=a$ and $ae_2+b=b$

$e_1=1$ and $e_2=0$ since $a \neq 0$

The identity element is (1,0)

c) Let the inverse of (a,b) be (c,d) if it exists

Then $(a,b)*(c,d)=(1,0)$ Ie.
$(ac,ad+b)=(1,0)$ $ac=1$ and
$ad+b=0$

ie. $c=1/a$ and $d=-b/a$, if $a \neq 0$

Thus the element (a,b) has an inverse if $a \neq 0$ and its inverse is $(1/a, -b/a)$.

4. Lagranges Theorem:

Let G be a finite group of order 'n' and H be any subgroup of G. then the order of H divides the order of G. ie. O(H)/ O(G)

Proof:

Let (G,*) be a group whose order is n. O(G) = n

Let (H,*) be a subgroup of G whose order is m.

O(H) = m

Let $h_1, h_2, h_3, \ldots, hm$ be the 'm' different elements of H.

The right coset H *a of H in G is difined by

$H*a = \{h_1*a, h_2*a, \ldots, h_m*a\}$, $a \in G$.

Since there is a one-one correspondence between the elements of H and H*a, the elements of H*a are distinct.

Hence each right coset of H in G has m distinct elements.

We know that any right cosets of H in G are either disjoint or identical.

The number of distinct right cosets of H in G is finite (say K). [G is finite]

The union of these K distinct cosets of H in G is equal to G.

Let these K distinct right cosets be

$H*a_1, H*a_2, H*a_3, \ldots, H*a_k$

Then $G = (H*a_1) U (H*a_2) U, \ldots, U (H*a_k)$

$O(G) = O(H*a_1) + (H*a_2) +, \ldots, + O(H*a_k)$

N = m + m +….. + m (k times)

N = km

$\Rightarrow k = \frac{n}{m}$ ie. $\frac{n}{m} = k$ ie. $\frac{O(G)}{O(m)} = k$

since k is an integer (time), m is a divisor of n.

$\Rightarrow$O(H) is a divisor of O(G)

$\Rightarrow$O(H) divides O(G). This proves the Lagrange's theorem.

UNIT-5

LATTICES AND BOOLEAN ALGEBRA

PART-A

1. Prove that $a + \overline{a}b = a + b$

Soln:

L.H.S= $a + \overline{a}b$

$= a + ab + \overline{a}b$ (since a=a+ab)

$= a + b(a + \overline{a})$

=a+b(1)

$a + \overline{a}b$ =a+b

2. Define Lattice .

A Lattice is a partially ordered set (poset)(L, $\leq$), in which for every pair of elements a,b ЄL, both greatest lower bound (GLB)and least upper bound (LUB)exist.

3. Define an equivalence relation

Let a be any set R be a relation defined on X. If R satisfies Reflexive, symmetric and transitive then the relation R is said to be an equivalence relation.

4. Let (L,$\wedge$,$\vee$) be a lattice . Then for any a,b,c $\in$ L, a$\wedge$ a=a and a$\vee$a=a.

Proof:

a$\vee$a = LUB(a,a) = LUB(a) = a

a$\vee$a=a

Now a$\wedge$a = GLB(a,a) = GLB(a) = a

a$\wedge$a=a.

5. Let a,b,c $\in$B. Show that (1) a.0=0 (2) a+1=1.

Solution:

a.0 = (a.0)+0

= (a.0)+(a.a$'$)

$= a.(0.+ a')$

$= a. a'$

$=0.$

By taking dual of a.0= 0, we have a+1=1.

6. Show that in a Boolean algebra the law of the double complements holds.

Soln:

It is enough to show that $a+a' =1$ and $a.a' =0$

By domination laws of Boolean Algebra we get

$a+a' =1$ and $a.a' =0$

By commutative laws we get $a' +a =1$ and $a' .a=0$

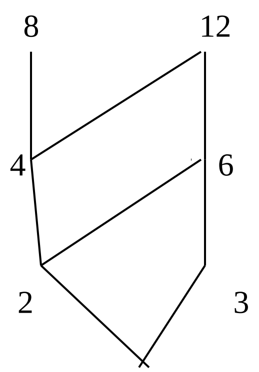
Therefore complement of a' is a

$(a')' =a$

7. Draw the hasse diagram for {(a,b)/a divides b} on{1,2,3,4,6,8,12}.

Soln:

The Relation R is
{(1,2)(1,3)(1,4)(1,6)(1,8)(1,12)(2,4)(2,6)(2,8)(2,12)(3,6)(3,12)(4,8)(4,12)(6,12)}



8. Show that in a Lattice if $a \leq b$ and $c \leq d$, then $a \wedge c \leq b \wedge d$.

Soln:

Given $a \leq b \Rightarrow a \wedge b=a$

$c \leq d \Rightarrow c \wedge d=c$

Claim: a c b d.

$(a \wedge c) \wedge (b \wedge d) = a \wedge c$

Now , L.H.S$=(a \wedge c) \wedge (b \wedge d)$

$= a \wedge (c \wedge b) \wedge d$

$= a \wedge (b \wedge c) \wedge d$

$= (a \wedge b) \wedge (c \wedge d)$

$(a \wedge c) \wedge (b \wedge d) = a \wedge c.$

$\Rightarrow a \wedge c \leq b \wedge d.$

9. Prove that any Lattice homomorphism is order preserving.

Proof:

Let $f: L_1 \to L_2$ be a homomorphism.

Let $a \leq b$

Then GLB $\{a,b\} = a \wedge b = a$

LUB$\{a,b\} = a \vee b = b$

Now $f(a \wedge b) = f(a)$

$f(a) \wedge f(b) = f(a)$

GLB $\{f(a), f(b)\} = f(a)$

Therefore, $f(a) \leq f(b)$

If $a \leq b \Rightarrow f(a) \leq f(b)$

Therefore, f is order preserving.

10. Define bounded lattice :

Let$(L, \wedge, \vee)$ be a given lattice. If it has both '0' element and '1' element then it is said to be bounded lattice. It is denoted by $(L, \wedge, \vee 0,1)$.

PART-B

1. If R is the relation on the set of integers such that (a,b) $\in$R if and only if $b=a^m$ for some positive integer m, show that R is a partial ordering.

Soln:

Since $a=a^1$, we have (a,a) $\in$R

Therefore R is reflexive ,

Let (a,b) $\in$R and (b,a) $\in$R

$b=a^m$ and $a=b^n$ -------------------- (1)

where m and n are positive integers.

$a=(b)^n=(a^m)^n$

$=a^{mn}$

Which means mn=1 or a=1 or a=-1 using (1)

Case(i):if mn = 1 then m=1 and n=1.

Therefore a=b

Case(ii): if a=1, then from (1),

$b=1^m=1=a$

if b=1, then from(1)

$a=1^n=1=b$

therefore a=b

Case(3):

If a=-1, then b=-1

Therefore a=b,

In all 3 cases, a=b

therefore, R is antisymmetric.

Let (a,b) $\in$R and (b,c) $\in$R

i.e, $b= a^m$ and $c=b^n$

$c = b^n = (a^m)^n = a^{mn}$

$c = a^{mn}$

therefore $(a,c) \in R$,

therefore R is transitive.

R is a partial order relation.

2. Let R be a relation on a set A. then define $R^{-1} = \{(a,b) \in AXA / (b,a) \in R\}$**.Prove that if (A,R) is poset then (A,R^{-1}) is also a poset.**

Soln:

Given A is finite set.

$R=\{(a,b)\}$ is a partial order relation on A.

Claim:

$R^{-1}=\{(a,b)\}$ is a partial order relatin.

Since $(a,a) \in R$

$R^{-1}=\{(a,b)\}$ is a reflexive.

Given R is antisymmentric.

$(a,b) \in R$ and $(b,a) \in R \Rightarrow a=b$

Since $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

$(b,a) \in R \Rightarrow (a,b) \in R^{-1}$

$(a,b) \in R^{-1}$ and $(b,a) \in R^{-1} \Rightarrow a=b$

Therefore $R^{-1}$is antisymmentric.

Given R is transitive,

$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

Since $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

$(b,c) \in R \Rightarrow (c,b) \in R^{-1}$

$(c,b) \in R^{-1}$ and $(b,a) \in R^{-1} \Rightarrow (c,a) \in R^{-1}$

Therefore R^{-1} is transitive.

Therefore R^{-1} is partial order relation.

Therefore , (A,R^{-1}) is a poset.

3. State and prove distribute inequality of lattice.

Statement :

Let $(L, \wedge, \vee)$ be a given lattice for any $a,b,c \in L$, the following in equality holds.

(i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Proof:

Claim-1: $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

From the definition of LUB , it is obvious that

$a \leq a \vee b$ (1)

and $b \wedge c \leq b \leq a \vee b$

$\Rightarrow b \wedge c \leq a \vee b$ (2)

From (1)& (2) , $a \vee b$ is an upper bound of { a, $b \wedge c$ }

Hence $a \vee b \geq a \vee (b \wedge c)$ (A)

From the definition of LUB , it is obvious that

$a \leq a \vee c$ (3)

and $b \wedge c \leq c \leq a \vee c$

$\Rightarrow b \wedge c \leq a \vee c$ (4)

From (3)& (4) , $a \vee c$ is an lower bound of { a, $b \wedge c$ }

Hence $a \vee c \geq a \vee (b \wedge c)$ (B)

From (A) &(B) , we have $a \vee (b \wedge c)$ is a lower bound of { $a \vee b$, $a \vee c$ }

Therefore

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

Hence the proof (i).

Claim -2:

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

We know that ,

$$a \geq (a \wedge b) \qquad (1)$$

and $b \vee c \geq b \geq (a \wedge b)$

$$\Rightarrow b \wedge c \geq (a \wedge b) \qquad (2)$$

From (1)& (2) , $(a \wedge b)$ is an upper bound of { a, $b \vee c$ }

Hence $a \wedge b \leq a \wedge (b \vee c)$ (c)

From the definition of LUB , it is obvious that

$$a \geq a \wedge c \qquad (3)$$

and $b \vee c \geq c \geq a \wedge c$

$$\Rightarrow b \vee c \geq a \wedge c \qquad (4)$$

From (3)& (4) , $a \vee c$ is an lower bound of { a, $b \wedge c$ }

Hence $a \wedge c \leq a \wedge (b \vee c)$ (D)

From (C) &(D) , we have $\wedge$ $a \wedge (b \vee c)$ is a upper bound of {$a \wedge b$, $a \vee c$ }

Therefore $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Hence the proof (ii).

4. In a Complemented , distributive lattice, show that the following are equivalent.

$\mathbf{a \leq b} \Leftrightarrow \quad a \wedge b' \mathbf{=0} \quad \Leftrightarrow a' \vee b \quad \mathbf{=1} \quad \Leftrightarrow \quad b' \leq a'$

Solution:

(i) $\Rightarrow$(ii)

Let $a \leq b$

Then $a \wedge b =$ a and $a \vee b =$ b (1)

$a \wedge b' = ((a \wedge b) \wedge b')$ using (1)

$=(a \wedge b \wedge b')$ (associative law)

$= a \wedge 0$

$=0.$

Hence $a \leq b \Rightarrow \quad a \wedge b' =0$

(ii) $\Rightarrow$(iii)

Let $a \wedge b' =0$

Take complement on both sides, we have

$(a \wedge b')' = (0)'$

$a' \vee b =1$

$\therefore \; a \wedge b' =0 \Rightarrow a' \vee b =1$

(iii) $\Rightarrow$(iv)

Let $a' \vee b =1$

$\Rightarrow (a' \vee b) \wedge b' = 1 \wedge b'$ (by cancellation law)

$\Rightarrow (a' \wedge b') \vee (b \wedge b') = b'$

$\Rightarrow (a' \wedge b') \vee 0 = b'$

$\Rightarrow (a' \wedge b') = b'$

$\therefore\ b' \leq a'$

$\therefore\ a' \vee b\ =1 \quad \Rightarrow \quad b' \leq a'$

(iv) $\Rightarrow$(i)

Let $b' \leq a'$

Then $(a' \wedge b') = b'$

Take complement on both side s

$(a' \wedge b')' = (b')'$

$\Rightarrow a \vee b\ =$b

$\Rightarrow$b $\geq$ a

Or a $\leq$ b

$b' \leq a' \Rightarrow$ a $\leq$ b.

5. In any Boolean algebra, show that a=b iff $a\bar{b} + \bar{a}b = 0$

Soln: Let (B,.,+,0,1) be any Boolean Algebra.

Let a,b $\in$B and a=b (1)

Claim: $a\bar{b} + \bar{a}b = 0$

Now, $a\bar{b} + \bar{a}b = a\,.\bar{b} + \bar{a}.b$

$= a..\bar{a} + \bar{a}.a$

=0+0 $((\because\ a.a' = 0))$

$a\bar{b} + \bar{a}b$ =0

Conversely, Assume $a\bar{b} + \bar{a}b$ =0

$\Rightarrow$ a+ $a\bar{b} + \bar{a}b$ =a (Left cancelation law)

$\Rightarrow$ a+ $a\bar{b}$ =a (Absorption law)

$\Rightarrow (a + \bar{a}).(a + b) = a$ (distributive law)

$\Rightarrow 1.(a + b) = a$

$\Rightarrow (a + b) = a$

Consider, $a\bar{b} + \bar{a}b = 0$

$\Rightarrow a\bar{b} + \bar{a}b + b = b$ (Right Cancelation law)

$\Rightarrow a\bar{b} + b = b$ (Absorption law)

$\Rightarrow (a + b).(b + \bar{b}) = b$ (distributive law)

$\Rightarrow (a + b).1 = b$

$\Rightarrow$a+b=b